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PREFACE

Sir,
An equation has no meaning, for me unless it expresses a thought of GOD — *Ramanujam*
[Statement to a friend]

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers,

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **Sura's Mathematics Guide Volume I and Volume II** for +2 Standard – Edition 2023-24. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises and in-text questions in addition to precise answers for textual questions.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

Subash Raj, B.E., M.S.
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CHAPTER 1

APPLICATIONS OF MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

- If $|A| \neq 0$, then A is a non-singular matrix and if $|A| = 0$, then A is a singular matrix.
- The adjoint matrix of A is defined as the transpose of the matrix of co-factors of A.
- If $AB = BA = I_n$, then the matrix B is called the inverse of A.
- If a square matrix has an inverse, then it is unique.
- A^{-1} exists if and only if A is non-singular.
- Singular matrix has no inverse.
- If A is non – singular and AB = AC, then B = C (left cancellation law).
- If A is non – singular and BA = CA then B = C (Right cancellation law).
- If A and B are any two non-singular square matrices of order n, then adj (AB) = (adj B) (adj A)
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- Two matrices A and B of same order are said to the **equivalent** if one can be obtained from the other by the applications of elementary transformations (A~B).
- A non – zero matrix is in a **row - echelon f**orm if all zero rows occur as bottom rows of the matrix and if the first non – zero element in any lower row occurs to the right of the first non – zero entry in the higher row.
- The **rank** of a matrix A is defined as the order of a highest order non – vanishing minor of the matrix A [ρ(A)].
- The **rank** of a non – zero matrix is equal to the number of non – zero rows in a row – echelon form of the matrix.
- An **elementary matrix** is a matrix which is obtained from an identity matrix by applying only one elementary transformation. Every non-singular matrix can be transformed to an identity matrix by a sequence of elementary row operations.
- A system of linear equations having atleast one solution is said to be **consistent**.
- A system of linear equations having no solutions is said to be **inconsistent**.

IMPORTANT FORMULAE TO REMEMBER

- Co – factor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}
- For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_n$
  $AA^{-1} = A^{-1}A = I_n$
- If A is non – Singular then
  (i) $|A^{-1}| = \dfrac{1}{|A|}$
  (ii) $(A^T)^{-1} = (A^{-1})^T$
  (iii) $(\lambda A^{-1}) = \dfrac{1}{\lambda} A^{-1}$ where λ is a non – zero scalar.

Reversal law for inverses :

- $(AB)^{-1} = B^{-1}A^{-1}$ where A, B are non – singular matrices of same order.

Law of double inverse :

- If A is non - singular, A^{-1} is also non – singular and $(A^{-1})^{-1} = A$.
- If A is a non - singular square matrix of order n, then
  (i) $(\text{adj } A)^{-1} = \text{adj } (A^{-1}) = \dfrac{1}{|A|}\cdot A$
  (ii) $|\text{adj } A| = |A|^{n-1}$
  (iii) $\text{adj } (\text{adj } A) = |A|^{n-2}A$
  (iv) $\text{adj } (\lambda A) = \lambda^{n-1}\text{adj } (A)$ where λ is a non – zero scalar
  (v) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$
  (vi) $(\text{adj } A)^T = \text{adj } (A^T)$
- If a matrix contains at least one non – zero element, then $\rho(a) \geq 1$.
- The rank of identity matrix I_n is n.
  If A is an $m \times n$ matrix then $\rho(A) \leq \min\{m, n\}$.
- A square matrix A of order n is invertible if and only if $\rho(A) = n$.
- Transforming a non-singular matrix A to the form I_n, by applying row operations is called Gauss – Jordan method.

Matrix – Inversion method :

- The solution for $AX = B$ is $X = A^{-1}B$ where A and B are square matrices of same order and non – singular

Cramer's Rule :

- If $\Delta = 0$, Cramer's rule cannot be applied $x_1 = \dfrac{\Delta_1}{\Delta}, x_2 = \dfrac{\Delta_2}{\Delta}, x_3 = \dfrac{\Delta_3}{\Delta}$

Gaussian Elimination method :

Transform the augmented matrix of the system of linear equations into row – echelon form and then solve by back substitution method.

Rouches capelli Theorem :

A system of equations $AX = B$ is consistent if and if $\rho(A) = \rho([A|B])$

(i) If $\rho(A) = \rho([A|B]) = n$, the number of unknowns, then the system is consistent and has a unique solution.

(ii) If $\rho(A) = \rho([A|B]) = n-k$, $k \neq 0$ then the system is consistent and has infinitely many solutions.

(iii) If $\rho(A) \neq \rho([A|B])$, then the system is inconsistent and has no solution.

Homogeneous system of linear equations :

(i) If $\rho(A) = \rho([A|B]) = n$, then the system has a unique solution which is the trivial solution for trivial solution, $|A| \neq 0$

(ii) If $\rho(A) = \rho([A|B]) < n$, the system has a non – trivial solution.

For non – trivial solution, $|A| = 0$.

✦ $A^{-1} = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj } A$ ✦ $A = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj (adj } A)$

EXERCISE 1.1

1. Find the adjoint of the following :

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ **(iii)** $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Sol. **(i)** Let A $= \begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

adj A $= \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$

[Interchange the elements in the leading diagonal and change the sign of the elements in off diagonal]

(ii) Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$$\text{adj } A = \begin{pmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{bmatrix} +(8-7) - (6-3) + (21-12) \\ -(6-7) + (4-3) - (14-9) \\ +(3-4) - (2-3) + (8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $\lambda = \frac{1}{3}$

Since adj $(\lambda A) = \lambda^{n-1}$(adj A)

we get adj $\left(\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\right) = \left(\frac{1}{3}\right)^2$

adj $\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

∴ Required adjoint matrix

$$= \frac{1}{9}\begin{bmatrix} +\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} (2+4) & -(-4-2) & +(4-1) \\ -(4+2) & +(4-1) & -(-4-2) \\ +(4-1) & -(4+2) & +(2+4) \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{3}{9}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

[Taking 3 common from each entry]

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse (if it exists) of the following :

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Sol. (i) Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Since A is non – singular, A^{-1} exists

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{Now, adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

[Inter change the entries in leading diagonal and change the sign of elements in the off diagonal]

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

(ii) Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Expanding along R_1,

$$|A| = 5\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$

$$= 5(25-1) - 1(5-1) + 1(1-5)$$

$$= 5(24) - 1(4) + 1(-4)$$

$$= 120 - 4 - 4 = 120 - 8 = 112 \neq 0$$

Since A is non singular, A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(25-1) - (5-1) + (1-5) \\ -(5-1) + (25-1) - (5-1) \\ +(1-5) - (5-1) + (25-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

Taking 4 common from every entry we get,

$$\text{adj } A = 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{112} \cdot 4 \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$= \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Expanding along R_1 we get,

$$|A| = 2\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$

$$= 2(8-7) - 3(6-3) + 1(21-12)$$

$$= 2(1) - 3(3) + 1(9)$$

$$= 2 - \cancel{9} + \cancel{9} = 2 \neq 0$$

Since A is a non-singular matrix, A^{-1} exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(8-7) - (6-3) + (21-12) \\ -(6-7) + (4-3) - (14-9) \\ +(3-4) - (2-3) + (8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. If F (α) = $\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1} = F(-\alpha)$ [Hy - 2019; FRT - 2022]

Sol. Given that $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$.

Expanding along R_1 we get,

$$|F(\alpha)| = \cos\alpha \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} - 0 + \sin\alpha \begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix}$$

$= \cos\alpha\,(\cos - 0) + \sin\alpha\,(0 + \sin\alpha)$

$= \cos^2 + \sin^2\alpha = 1 \neq 0$

Since F (α) is a non-singular matrix, $[F(\alpha)]^{-1}$ exists.

Now, adj (F(α)) =

$$\begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ \sin\alpha & \cos\alpha \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & \sin\alpha \\ 0 & \cos\alpha \end{vmatrix} & +\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} & -\begin{vmatrix} \cos\alpha & 0 \\ -\sin\alpha & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & \sin\alpha \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(\cos\alpha - 0) & -(0) & +(0+\sin\alpha) \\ -(0) & +(\cos^2\alpha + \sin^2\alpha) & -(0) \\ +(0 - \sin\alpha) & -(0) & +(\cos - 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & 0 & +\sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\therefore F(\alpha)^{-1} = \frac{1}{|F(\alpha)|} \text{adj } (F(\alpha))$$

$$[F(\alpha)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad ...(1)$$

Now, $F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad ... (2)$$

[∵ cosα is an even function, cos (–α) = cos α and sin α is an odd function, sin (–α) = –sinα]

From (1) and (2)

$[F(\alpha)]^{-1} = F(-\alpha)$

Hence proved.

4. If A=$\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}.

Sol. Given $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$\therefore A^2 - 3A - 7I_2$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9+0 \\ -3+3+0 & 1+6-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$$

Hence proved.

$\therefore A^2 - 3A - 7I_2 = 0$

Post – multiplying by A^{-1} we get,

$A^2 . A^{-1} - 3AA^{-1} - 7I_2\,A^{-1} = 0.A^{-1}$

$\Rightarrow \quad A(AA^{-1}) - 3\,(AA^{-1}) - 7(A^{-1}) = 0$

[∵ $I_2\,A^{-1} = A^{-1}$ and $(0)A^{-1} = 0$]

$\Rightarrow \quad AI - 3I - 7A^{-1} = 0$ [∵ $AA^{-1} = I$]

$\Rightarrow \quad AI - 3I = 7A^{-1}$

$\Rightarrow \quad A^{-1} = \frac{1}{7}[A - 3I]$ [∴ AI = A]

$$\Rightarrow \quad A^{-1} = \frac{1}{7}\left[\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{7}\begin{bmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

CHAPTER 2

COMPLEX NUMBERS

MUST KNOW DEFINITIONS

- If a Complex number is of the form $x + iy$ where x is a real part and y is the imaginary part of the complex number.
- $z_1 = z_2$ if Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2)

Properties of complex numbers:

Under Additions

- Let z_1, z_2 and z_3 are complex numbers.
  - (i) Closure property ($z_1 + z_2$ is a complex number)
  - (ii) Commutative property ($z_1 + z_2 = z_2 + z_1$)
  - (iii) Associative property $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
  - (iv) Additive identity ($z + 0 = 0 + z = z$)
  - (v) Additive inverse $(z + -z) = (-z + z) = 0$
- **Under Multiplication:**
  - (i) Closure property ($z_1 z_2$ is also a complex number)
  - (ii) Commutative property ($z_1 z_2 = z_2 z_1$)
  - (iii) Associative property $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
  - (iv) Multiplicative identity ($z_1 . 1 = 1. z_1 = z_1$)
  - (v) Multiplicative inverse $z.w = w.z = 1 \Rightarrow w = z^{-1}$
- **Distributive property (Multiplication distributes over addition)**

  $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

  Also, $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$
- Conjugate of $x + iy$ is $x - iy$
- If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$
- $|z - z_0| = r$ is the equation of circle where z_0 is a fixed complex number and r is the distance from z_0 to z.
- Polar form of $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$

  where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.

De Moivre's theorems

- Given any complex number $\cos\theta + i\sin\theta$ and any integer n, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- $z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$, $k = 0, 1, 2, \ldots\ldots n-1$.

IMPORTANT FORMULA TO REMEMBER

- $i^0 = 1, i^1 = i, i^4 = 1, i^{-4} = 1, (i)^{-1} = -i, (i)^{-2} = -1, (i)^{-3} = i$
- $\sqrt{ab} = \sqrt{a}.\sqrt{b}$ is valid only if atleast one of a, b is non-negative.
- When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

  $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

  $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

  $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Properties of complex conjugates

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(3) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

(5) $\text{Re}(z) = \frac{z+\overline{z}}{2}$

(6) $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

(7) $\overline{\left(z^n\right)} = \left(\overline{z}\right)^n$, where n is an integer.

(8) z is real iff $z = \overline{z}$

(9) z is purely imaginary iff $z = -\overline{z}$

(10) $\overline{\overline{z}} = z$

Properties of modulus of a complex number

(1) $|z| = |\overline{z}|$

(2) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)

(3) $|z_1 z_2| = |z_1||z_2|$

(4) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(5) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(6) $|z^n| = |z|^n$, where n is an integer

(7) $\text{Re}(z) \leq |z|$

(8) $\text{Im}(z) \leq |z|$

- $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- $|z_1 - z_2| \leq |z_1| + |z_2|$
- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
- $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$
- $|z_1 z_2| = |z_1||z_2|$

- Let $a + ib = \sqrt{x+iy}$ then
  $x = \pm\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}}$, $y = \pm\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}$
- $|z-z_0| < r$ represents the points interior of the circle.
- $|z-z_0| > r$ represents the points exterior of the circle.
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- $\arg(z^n) = n \arg z$
- Alternate form of $\cos\theta + i\sin\theta$ is $\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)$, $k \in \mathbb{Z}$

Euler's formula

- $e^{i\theta} = \cos\theta + i\sin\theta$ or $z = re^{i\theta}$
- If $z = r(\cos\theta + i\sin\theta)$ then
  $z^{-1} = \dfrac{1}{r}(\cos\theta - i\sin\theta)$
- $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$
- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ and $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$
- $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$ and $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$
- $1 + \omega + \omega^2 + \ldots\ldots + \omega^{n-1} = 0$
  where ω is the n^{th}--- root of unity.
  $1\omega.\omega^2 \ldots.\omega^{n-1} = (-1)^{n-1}$
  $\omega^{n-k} = \omega^{-k} = (\bar{\omega})^k \quad 0 \le k \le n-1$

EXERCISE 2.1

Simplify the following:

1. $i^{1947} + i^{1950}$ **2.** $i^{1948} - i^{-1869}$

3. $\sum_{n=1}^{12} i^n$ **[FRT & May - 2022]**

4. $i^{59} + \dfrac{1}{i^{59}}$ **5.** $i\, i^2\, i^3 \ldots\, i^{2000}$

6. $\sum_{n=1}^{10} i^{n+50}$

Sol. 1. $i^{1947} + i^{1950} = i^{1944}.\, i^3 + i^{1948}.\, i^2$

[$\because$ 1944 is a multiple of 4 and1948 is also a multiple of 4]

$= (i^4)^{486}\,.\, i^2\,.\, i^1 + (i^4)^{487}\,.\, i^2$ [$i^4 = 1$]

$= (1^{486})(-1)(i) + (1)^{487}(-1)$ [$i^2 = -1$]

$= -i - 1 = -1 - i$

2. $i^{1948} - i^{-1869} = (i^4)^{487} - [i^{-1868}\,.\, i^{-1}]$

$= 1^{487} - \left[(i^4)^{-467}\cdot\dfrac{1}{i}\right]$ $[\because i^4 = 1]$

$= 1 - [1.(-i)]$ $[\because i^{-1} = \dfrac{1}{i} = -i]$

[One power any number is 1]

$= 1 + i$

3. $\sum_{n=1}^{12} i^n = (i^1 + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + (i^9 + i^{10} + i^{11} + i^{12})$

$= (i - 1 - i + 1) + (i^{4+1} + i^{4+2} + i^{4+3} + (i^4)^2) + (i^{8+1} + i^{8+2} + i^{8+3} + (i^4)^3)$

$= 0 + (i + i^2 + i^3 + i^4) + (i^1 + i^2 + i^3 + i^4)$

$[\therefore i^2 = -1, i^3 = -i, i^4 = 1]$

$= 0 + (i - 1 - i + 1) + (i - 1 - i + 1)$

$= 0 + 0 + 0 = 0$

4. $i^{4\times14+3} + i^{-(4\times14+3)}$ **[Hy - 2019; FRT - 2022]**

$= (i^4)^{14} . i^3 + (i^4)^{-14} . i^{-3}$

$= 1. i^3 + 1.i^{-3}$ $[\because i^4 = 1]$

$= -i + i$ $[\because i^3 = -i \text{ and } i^{-3} = i]$

$= 0.$

5. $i. i^2\ i^3 \ ......i^{2000} = i^{1+2+3+ \,......+2000}$

$= i^{\frac{2000\times2001}{2}}$ $\left[\because 1+2+3+\ldots n = \frac{n(n+1)}{2}\right]$

$= i^{1000\times2001} = i^{2001000} = 1$

[$\because$ 2001000 is divisible by 4 as its last two digits are divisible by 4]

6. $i^{1+50} + i^{2+50} + ...+ i^{10+50}$

$= i^{51} + i^{52} + ...+ i^{60}$

Taking i^{50} common we get,

$i^{50} [(i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + i^9 + i^{10}]$

$= i^{50} [0 + (i^{4+1} + i^{4+2} + i^{4+3} + i^{4+4}) + (i^{8+1} + i^{8+2})]$

$= i^{50} [0 + 0 + i + i^2]$ $[\because i + i^2 + i^3 + i^4 = 0]$

$= i^{50} [i - 1] = i^{48+2} (i - 1)$

$= i^2 (i - 1)$ $[\because i^{48} = 1]$

$= -1 (i - 1) = -i + 1 = 1 - i$

EXERCISE 2.2

1. Evaluate the following if $z = 5 - 2i$ and $w = -1+3i$

(i) $z + w$ **(ii)** $z - iw$

(iii) $2z + 3w$ **(iv)** $z\,w$

(v) $z^2+2zw+w^2$ **(vi)** $(z + w)^2$

Sol. **(i)** $z + w$

$= (5 - 2i) + (-1 + 3i) = (5 - 1) + i (-2 + 3)$

$= 4 + i (1) = 4 + i$

(ii) $z - iw$

$= (5 - 2i) - i (-1 + 3i)$

$= (5 - 2i) + (+i - 3i^2)$

$= 5 - 2i + i - 3 (-1) = 5 - i + 3 = 8 - i$

(iii) $2z + 3w$

$= 2 (5 - 2i) + 3 (-1 + 3i) = 10 - 4i - 3 + 9i$

$= (10 - 3) + i (-4 + 9) = 7 + 5i$

(iv) $z\,w$

$= (5 - 2i) (-1 + 3i) = -5 + 15i + 2i - 6i^2$

$= -5 + 17i - 6 (-1) = -5 + 17i + 6 = 1 + 17i$

(v) $z^2 + 2zw + w^2$

$= (5 - 2i)^2 + 2 (5 - 2i) (-1 + 3i) + (-1 + 3i)^2$

$= 25 + 4i^2 - 20i + 2[-5 + 15i + 2i - 6i^2] + 1 + 9i^2 - 6i$

$= 25 - 4 - 20i + 2 (-5 + 17i + 6) + 1 - 9 - 6i$ $[\because i^2 = -1]$

$= 21 - 20i + 2 (1 + 17i) - 8 - 6i$

$= 21 - 20i + 2 + 34i - 8 - 6i = 15 + 8i$

(vi) $(z + w)^2$

$= [(5 - 2i) + (-1 + 3i)]^2 = (4 + i)^2$

$= 16 + 8i - 1 = 16 - 1 + 8i = 15 + 8i$

2. Given the complex number $z = 2 + 3i$, represent the complex numbers in Argand diagram.

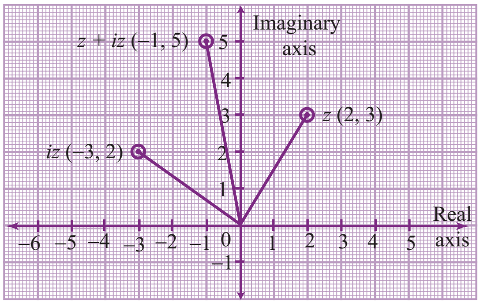
(i) z, iz, and $z + iz$ **(ii) z, $-iz$, and $z - iz$.**

Sol. **(i)** Represent z, iz and $z + iz$ in the Argand diagram.

$z = 2 + 3i$ can be represented as (2, 3)

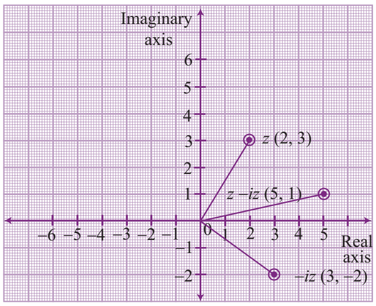
$iz = i (2 + 3i) = 2i + 3i^2 = 2i - 3 = -3 + 2i$ can be represented as (–3, 2)

$z + iz = 2 + 3i - 3 + 2i = -1 + 5i$ can be represented as (–1, 5) in the argand diagram.



(ii) $z = 2 + 3i$ can be represented as (2, 3) $-iz = -i (2 + 3i) = -2i - 3i^2 = -2i - 3 (-1) = 3 - 2i$

$z - iz = 2 + 3i + 3 - 2i = 5 + \text{i}$ can be represented as (5, 1) in the Argand place.



3. Find the values of the real numbers x and y, if the complex numbers $(3 - i)x - (2 - i)y + 2i + 5$ and $2x + (-1 + 2i)y + 3 + 2i$ are equal **[Hy - 2019]**

Sol. Given $(3 - i)x - (2 - i)y + 2i + 5$

$= 2x + (-1 + 2i)y + 3 + 2i$

$\Rightarrow 3x - ix - 2y + iy + 2i + 5 = 2x - y + 2iy + 3 + 2i$

choosing the real and imaginary parts

$(3x - 2y + 5) + i(-x + y + 2) = 2x - y + 3 + i(2y + 2)$

Equating the real and imaginary parts both sides, we get

$$3x - 2y + 5 = 2x - y + 3$$

$$\Rightarrow 3x - 2y + 5 - 2x + y - 3 = 0$$

$$\Rightarrow x - y = -2 \quad \ldots (1)$$

$$-x + y + 2 = 2y + 2$$

$$\Rightarrow -x + y + 2 - 2y - 2 = 0$$

$$\Rightarrow -x - y = 0 \Rightarrow x + y = 0 \quad \ldots (2)$$

(1) – (2) we get,

$$x - \cancel{y} = -2$$
$$x + \cancel{y} = 0$$
$$2x = -2$$

$$\Rightarrow x = -1$$

Substituting $x = -1$ in (2) we get,

$$-1 + y = 0 \Rightarrow y = 1$$

$\therefore x = -1$ and $y = 1$

EXERCISE 2.3

1. If $z_1 = 1 - 3i$, $z_2 = -4i$ and $z_3 = 5$, show that

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Sol. (i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

Given $z_1 = 1 - 3i$, $z_2 = -4i$ and $z_3 = 5$

$$\text{LHS} = (z_1 + z_2) + z_3 = [1 - 3i + (-4i)] + 5 = [1 - 7i] + 5 = 6 - 7i$$

$$\text{RHS} = z_1 + (z_2 + z_3) = 1 - 3i + (-4i + 5) = 6 - 7i$$

LHS = RHS

$$\therefore (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

$$\text{LHS} = (z_1 z_2) z_3$$
$$= [(1 - 3i)(-4i)] 5 = [-4i + 12i^2] 5$$
$$= (-4i - 12) 5 = -20i - 60$$

$$\text{RHS} = z_1 (z_2 z_3)$$
$$= (1 - 3i)[(-4i) 5] = (1 - 3i)(-20i)$$
$$= -20i + 60i^2 = -20i - 60$$

$$\text{LHS} = \text{RHS}$$

$$\therefore (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

2. If $z_1 = 3$, $z_2 = 7i$, and $z_3 = 5 + 4i$, show that

(i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ **[July - 2022]**

(ii) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

Sol. (i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Given $z_1 = 3$, $z_2 = -7i$, $z_3 = 5 + 4i$

$$\text{LHS} = z_1 (z_2 + z_3) = 3[-7i + 5 + 4i] = 3[5 - 3i] = 15 - 9i$$

$$\text{RHS} = z_1 z_2 + z_1 z_3 = 3(-7i) + 3(5 + 4i) = -21i + 15 + 12i = -9i + 15 = 15 - 9i$$

LHS = RHS

$$\therefore z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

(ii) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

$$\text{LHS} = (z_1 + z_2) z_3 = (3 - 7i)(5 + 4i) = 15 + 12i - 35i - 28i^2 = 15 - 23i + 28 = 43 - 23i$$

$$\text{RHS} = z_1 z_3 + z_2 z_3 = 3(5 + 4i) + (-7i)(5 + 4i) = 15 + 12i - 35i - 28i^2 = 15 - 23i + 28 = 43 - 23i$$

$$\text{LHS} = \text{RHS}$$

$$\therefore (z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$$

3. If $z_1 = 2 + 5i$, $z_2 = -3 - 4i$, and $z_3 = 1 + i$, find the additive and multiplicative inverse of z_1, z_2 and z_3.

Sol. Given $z_1 = 2 + 5i$, $z_2 = -3 - 4i$ and $z_3 = 1 + i$

Additive inverse of z_1 is $-z_1 = -(2 + 5i) = -2 - 5i$

Additive inverse of z_2 is $-z_2 = -(-3 - 4i) = 3 + 4i$

Additive inverse of z_3 is $-z_3 = -(1 + i) = -1 - i$

Multiplicative inverse of z_1 is

$$\frac{1}{z_1} = \frac{1}{2 + 5i} \times \frac{2 - 5i}{2 - 5i}$$

[Multiply and divide by the conjugate of denominator]

$$= \frac{2-5i}{2^2-(5i)^2} = \frac{2-5i}{4-25i^2} = \frac{2-5i}{4+25}$$

$$= \frac{1}{29}(2-5i) \quad [\therefore i^2 = -1]$$

Multiplicative inverse of z_2 is

$$\frac{1}{z_2} = \frac{1}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{-3+4i}{(-3)^2-(4i)^2}$$

$$= \frac{-3+4i}{9-16i^2} = \frac{-3+4i}{9+16} = \frac{1}{25}(-3+4i)$$

Multiplicative inverse of z_3 is

$$\frac{1}{z_3} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{1^2-(i^2)} = \frac{1-i}{1+1} = \frac{1}{2}(1-i)$$

EXERCISE 2.4

1. Write the following in the rectangular form :

(i) $\overline{(5+9i)+(2-4i)}$

(ii) $\frac{10-5i}{6+2i}$ **(iii)** $\overline{3i}+\frac{1}{2-i}$

Sol. (i) $\overline{(5+2)+(9i-4i)} = \overline{7+5i}$

$= 7-5i$ [$\because$ Conjugate of $7+5i$ is $7-5i$]

(ii) $\frac{10-5i}{6+2i} \times \frac{6-2i}{6-2i}$

[Multiply and divide by the conjugate of the denominator]

$$= \frac{60-20i-30i+10i^2}{6^2-(2i)^2} = \frac{60-50i-10}{36+4}$$

$$= \frac{50-50i}{40} = \frac{50(1-i)}{40} = \frac{5}{4}(1-i)$$

(iii) $-3i+\frac{1}{2-i} \times \frac{2+i}{2+i}$

[$\because$ Conjugate of $3i$ is $-3i$]

$$= -3i+\frac{2+i}{2^2-i^2} = -3i+\frac{2+i}{4+1} = -3i+\frac{2+i}{5}$$

$$= \frac{-15i+2+i}{5} = \frac{-14i+2}{5} = \frac{2}{5}-\frac{14i}{5}$$

2. If $z = x+iy$, find the following in rectangular form.

(i) $\text{Re}\left(\frac{1}{z}\right)$ **(ii)** $\text{Re}(i\bar{z})$

(iii) $\text{Im}(3z+4\bar{z}-4i)$

Sol. (i) $\text{Re}\left(\frac{1}{x+iy}\right) = \text{Re}\left(\frac{1}{x+iy} \times \frac{x-iy}{x-iy}\right)$

$$= \text{Re}\left(\frac{x-iy}{x^2-(i^2y^2)}\right) = \text{Re}\left(\frac{x-iy}{x^2+y^2}\right)$$

$\therefore$ Real part is $\frac{x}{x^2+y^2}$

(ii) $\text{Re}(i(x-iy))$

[$\because$ When $z = x+iy$, $\bar{z} = x-iy$]

$= \text{Re}(ix-i^2y)$

$= \text{Re}(ix+y)$ [$\because i^2 = -1$]

$= \text{Re}(y+ix)$

$\therefore$ Real part is y

(iii) $\text{Im}(3(x+iy)+4(x-iy)-4i)$

$= \text{Im}(3x+3iy+4x-4iy-4i)$

$= \text{Im}(3x+4x+i)(3y-4y-4)$

$= \text{Im}(7x+i(-y-4))$

$\therefore$ Imaginary part is $-y-4$.

3. If $z_1 = 2-i$ and $z_2 = -4+3i$, find the inverse of $z_1 z_2$ and $\frac{z_1}{z_2}$. **[PTA - 5]**

Sol. Given $z_1 = 2-i$ and $z_2 = -4+3i$

$$\begin{aligned} z_1 z_2 &= (2-i)(-4+3i) \\ &= -8+6i+4i-3i^2 \\ &= -8+10i-3(-1) \\ &= -8+10i+3 = -5+10i \end{aligned}$$

Inverse of $z_1 z_2$ is $\frac{1}{z_1 z_2}$

$$= \frac{1}{-5+10i} \times \frac{-5-10i}{-5-10i} = \frac{-5-10i}{(-5)^2-(10i)^2}$$

$$= \frac{-5-10i}{25-100i^2} = \frac{-5-10i}{25+100} \quad [\because i^2 = -1]$$

$$= \frac{\cancel{5}(-1-2i)}{\cancel{5}(25)} = \frac{-1-2i}{25}$$

$\therefore$ Inverse of $z_1 z_2$ is $\frac{1}{25}(-1-2i)$

Inverse of $\frac{z_1}{z_2}$ is $\frac{1}{\frac{z_1}{z_2}} = \frac{z_2}{z_1}$

$$\therefore \text{Inverse of } \frac{z_1}{z_2} = \frac{z_2}{z_1} = \frac{-4+3i}{2-i} \times \frac{2+i}{2+i}$$

$$= \frac{-8-4i+6i+3i^2}{2^2-(i^2)}$$

$$= \frac{-8+2i-3}{4+1} = \frac{-11+2i}{5}$$

$$= \frac{1}{5}(-11+2i)$$

$\therefore$ Inverse of $\frac{z_1}{z_2}$ is $\frac{1}{5}(-11+2i)$

4. The complex numbers u, v and w are related by $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$. If $v = 3 - 4i$ and $w = 4 + 3i$, find u in rectangular form.

Sol. $$\frac{1}{u} = \frac{1}{v}+\frac{1}{w} = \frac{1}{3-4i}+\frac{1}{4+3i} = \frac{4+3i+3+4i}{(3-4i)(4+3i)}$$

$$= \frac{7-i}{12+9i-16i-12i^2} = \frac{7-i}{12-7i-12(-1)}$$

$$= \frac{7-i}{12-7i-12} = \frac{7-i}{24-7i}$$

$$u = \frac{24-7i}{7-i} \times \frac{7+i}{7+i}$$

$$= \frac{168-24i-49i-7i^2}{49+1} = \frac{168-25i+7}{50}$$

$$= \frac{175-25i}{50} = \frac{25(7-i)}{50} = \frac{(7-i)}{2}$$

$$u = \frac{1}{2}(7-i)$$

5. Prove the following properties:

(i) z is real if and only if $z = \bar{z}$

(ii) Re(z) = $\frac{z+\bar{z}}{2}$ and Im(z) = $\frac{z-\bar{z}}{2i}$ **[May - 2022]**

Sol. (i) Let $z = x + iy$

Then $\bar{z} = x - iy$

$$z = \bar{z}$$

$$\Leftrightarrow \quad x + iy = x - iy$$

$$\Leftrightarrow x + iy - x + iy = 0$$

$$\Leftrightarrow \quad 2iy = 0$$

$$\Leftrightarrow \quad y = 0$$

[$\because$ 2 and i are constants]

when $y = 0$, $z = x$ which is real.

$\therefore$ z is purely real $\Leftrightarrow z = \bar{z}$

(ii) Let $z = x + iy$ where x is the Re (z) and y is the Im (z).

Then $\bar{z} = x - iy$

$$z + \bar{z} = x + iy + x - iy = 2x$$

$$\therefore \frac{z+\bar{z}}{2} = x$$

$$\frac{z+\bar{z}}{2} = \text{Re}(z)$$

Also $z - \bar{z} = x + iy - (x - iy)$

$= x + iy - x + iy = 2iy$

$$\frac{z-\bar{z}}{2i} = y$$

$$\frac{z-\bar{z}}{2i} = \text{Im}(z)$$

6. Find the least value of the positive integer n for which $(\sqrt{3}+i)^n$. (i) real (ii) purely imaginary.

Sol. (i) Let $z = (\sqrt{3}+i)^n$

$$z = (\sqrt{3}+i)^n = \left[2\left(\frac{\sqrt{3}+i}{2}\right)\right]^n$$

$$= 2^n\left[\frac{\sqrt{3}}{2}+\frac{i}{2}\right]^n$$

[Multiply and divide by 2]

$$= 2^n\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]^n = 2^n\left[\cos n\frac{\pi}{6}+i\sin n\frac{\pi}{6}\right] \quad ...(1)$$

$$\bar{z} = 2^n\left[\cos n\frac{\pi}{6}-i\sin n\frac{\pi}{6}\right] \quad ...(2)$$

Since z is real, $z = \bar{z}$

$$\Rightarrow 2^n\left[\cos n\frac{\pi}{6}+i\sin n\frac{\pi}{6}\right] = 2^n\left[\cos n\frac{\pi}{6}-i\sin n\frac{\pi}{6}\right]$$

[From (1) and (2)]

$$\Rightarrow 2i\sin n\frac{\pi}{6} = 0 \Rightarrow \sin n\frac{\pi}{6} = 0$$

$$\Rightarrow \sin n\frac{\pi}{6} = \sin\pi \qquad [\because \sin\pi = 0]$$

$$\Rightarrow \frac{n\pi}{6} = \pi \Rightarrow \frac{n}{6} = 1 \Rightarrow n = 6$$

(ii) **Since z is purely imaginary**

$$z = -\bar{z}$$

$$\therefore 2^n\left[\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right] = -2^n\left[\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right]$$

[From (1) & (2)]

$$\Rightarrow \cos\frac{n\pi}{6}+\cancel{i\sin\frac{n\pi}{6}} = -\cos\frac{n\pi}{6}+\cancel{i\sin\frac{n\pi}{6}}$$

$$\Rightarrow 2\cos\frac{n\pi}{6} = 0$$

$$\Rightarrow \cos n\frac{\pi}{6} = 0 = \cos\frac{\pi}{2} \qquad [\because \cos\frac{\pi}{2} = 0]$$

$$\Rightarrow \frac{n\cancel{\pi}}{6} = \frac{\cancel{\pi}}{2} \Rightarrow n = \frac{6}{2}$$

$$\Rightarrow n = 3.$$

7. Show that

(i) $\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$ **is purely imaginary.** **[PTA-3; FRT & July - 2022]**

(ii) $\left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$ **is real.**

Sol. (i) Let $z = \left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$

Now $\bar{z} = \overline{\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}}$

$\bar{z} = \overline{\left(2+i\sqrt{3}\right)^{10}} - \overline{\left(2-i\sqrt{3}\right)^{10}}$ $[\because \overline{z_1-z_2} = \bar{z_1} - \bar{z_2}]$

$= \left(2-i\sqrt{3}\right)^{10}-\left(2+i\sqrt{3}\right)^{10}$

$= -\left[\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}\right] = -z$

$\therefore \bar{z} = -z \Rightarrow z$ is purely imaginary

Hence $\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$ is purely imaginary

(ii) Consider $\frac{19-7i}{9+i} = \frac{19-7i}{9+i}\times\frac{9-i}{9-i}$

$= \frac{171-19i-63i+7i^2}{(9)^2-i^2}$

$= \frac{171-82i-7}{81+1} = \frac{164-82i}{82}$

$= \frac{82(2-i)}{82} = 2-i$

Also $\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i}\times\frac{7+6i}{7+6i}$

$= \frac{140+120i-35i-30i^2}{7^2-(6i)^2}$

$= \frac{140+85i+30}{49+36} = \frac{170+85i}{85}$

$= \frac{\cancel{85}(2+i)}{\cancel{85}} = 2+i$

$\therefore \left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12} = (2-i)^{12}+(2+i)^{12}$

Let $z = (2-i)^{12}+(2+i)^{12}$

$\therefore \bar{z} = \overline{(2-i)^{12}+(2+i)^{12}}$

$= \overline{(2-i)^{12}}+\overline{(2+i)^{12}}$ $[\because \overline{z_1+z_2} = \bar{z_1}+\bar{z_2}]$

$= (2+i)^{12}+(2-i)^{12} = z$

$\therefore \bar{z} = z \Rightarrow z$ is purely real

$\therefore \left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$ is real

EXERCISE 2.5

1. Find the modulus of the following complex numbers

(i) $\frac{2i}{3+4i}$ **(ii)** $\frac{2-i}{1+i}+\frac{1-2i}{1-i}$

(iii) $(1-i)^{10}$ **(iv)** $2i(3-4i)(4-3i)$

Sol. (i) Let $z = \frac{2i}{3+4i}$

$|z| = \left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|} = \frac{\sqrt{2^2}}{\sqrt{3^2+4^2}} = \frac{2}{\sqrt{9+16}}$

$= \frac{2}{\sqrt{25}} = \frac{2}{5}$

(ii) Let $z = \frac{2-i}{1+i}+\frac{1-2i}{1-i}$

$= \frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}$

$= \frac{2-2i-i+i^2+1+i-2i-2i^2}{1^2-i^2}$

$= \frac{2-3i-1+1-i+2}{2} = \frac{4-4i}{2}$

$= \frac{2(2-2i)}{2} = 2-2i$

$\left|\frac{2-i}{1+i}+\frac{1-2i}{1-i}\right| = 2-2i$

$\therefore |z| = \sqrt{2^2+(-2)^2}$

$= \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

CHAPTER

3

THEORY OF EQUATIONS

MUST KNOW DEFINITIONS

- For the quadratic equation $ax^2 + bx + c = 0$,
  (i) $\Delta = b^2 - 4ac > 0$ iff the roots are real and distinct
  (ii) $\Delta = b^2 - 4ac < 0$ iff the equation has no real roots

Fundamental theorem of algebra :

- Every polynomial equation of degree n has at least one root in C.

Complex conjugate root theorem :

- If a complex number z_0 is a root of a polynomial equation with real co-efficients, then complex conjugate $\overline{z_0}$ is also a root.
- If $p + \sqrt{q}$ is a root of a quadratic equation then $p - \sqrt{q}$ is also a root of the same equation where p. q are rational and $\sqrt{q}$ is irrational.
- If $\sqrt{p} + \sqrt{q}$ is a root of a polynomial equation then $\sqrt{p} - \sqrt{q}$, $-\sqrt{p} + \sqrt{q}$, and $-\sqrt{p} - \sqrt{q}$ are also roots of the same equation.
- If the sum of the co-efficients in $p(x) = 0$. Then 1 is a root of $p(x)$.
- If the sum of the co-efficients of odd powers = sum of the co-efficients of even powers, then -1 is a root of $p(x)$.

Rational root theorem :

- Let $a_n x^n + ...+ a_1x + a_0$ with $a_n \neq 0$, $a_0 \neq 0$ be a polynomial with integer co-efficients. If $\frac{p}{q}$ with $(p, q) = 1$, is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n

Reciprocal polynomial :

- A polynomial $p(x)$ of degree n is said to be a reciprocal polynomial if one of the conditions is true

  (i) $p(x) = x^n \ p\left(\frac{1}{x}\right)$ (ii) $p(x) = -x^n. \ p\left(\frac{1}{x}\right)$
- A change of sign in the co-efficients is said to occur at the j^{th} power of x in $p(x)$ if the co-efficient of x^{j+1} and the co-efficient of x^j (or) co-efficient. of x^{j-1}, the co-efficient of x^j are of different signs.

IMPORTANT FORMULAE TO REMEMBER

- Vieta's formula for quadratic equation

  If α, β are the roots of $ax^2 + bx + c = 0$ then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

  Also, $x^2 - x$ (sum of the roots) + product of the roots = 0
- Vieta's formula for polynomial of degree 3.

  Co-efficient. of $x^2 = -(\alpha + \beta + \gamma)$ where α, β, γ are its roots

  Co-efficient of $x = \alpha\beta + \beta\gamma + \gamma\alpha$ and constant term $= -\alpha\beta\gamma$
- Vieta's formula for polynomial equation of degree $n > 3$

  Co-efficient of $x^{n-1} = \Sigma_1 = -\Sigma\alpha_1$

  Co-efficient of $x^{n-2} = \Sigma_2 = -\Sigma\ \alpha_1\alpha_2$

  Co-efficient of $x^{n-3} = \Sigma_3 = -\Sigma\ \alpha_1\alpha_2\alpha_3$

  Co-efficient of $x = \Sigma_{n-1} = (-1)^{n-1}\ \Sigma\ \alpha_1\alpha_2 \ldots \alpha_{n-1}$

  Co-efficient of x^0 = constant term $= \Sigma_n = (-1)^n\ \alpha_1\alpha_2 \ldots \alpha_n$

  A polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ $(a_n \neq 0)$ is a reciprocal equation iff one of the following statements is true.

  (i) $a_n = a_0, a_{n-1} = a_1, a_{n-2} = a_2, \ldots$

  (ii) $a_n = -a_0, a_{n-1} = -a_1, a_{n-2} = -a_2, \ldots$

Descartes rule:

- If p is the number of positive zeros of a polynomial $p(x)$ with real co-efficients and s is the number of sign changes in co-efficient of $p(x)$, then $s - p$ is a non negative even integer

EXERCISE 3.1

1. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid. **[Aug. - 2021]**

Sol. Let x be the side and V be the volume of the cube, then volume of the cube

$$V = x \times x \times x = x^3$$

If the sides are increased by 1, 2, 3 units, Volume of the cuboid is equal to volume of the cube increased by 52

$$V + 52 = (x+1) \times (x+2) \times (x+3)$$

If α, β, and γ are the roots, then If $\alpha = -1, \beta = -2$ and $\gamma = -3$.

$$\Sigma_1 = (\alpha + \beta + \gamma) = -1 - 2 - 3 = -6$$

$$\Sigma_2 = (\alpha\beta + \alpha\gamma + \beta\gamma) = (-1)(-2) + (-1)(-3) + (-2)(-3) = 2 + 3 + 6 = 11$$

$$\Sigma_3 = (\alpha\beta\gamma) = (-1)(-2)(-3) = -6$$

Hence, the volume of the cuboid equation is

$$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma^3 = x^3 - (-6)x^2 + (11)x - (-6) = x^3 + 6x^2 + 11x + 6$$

$$\therefore V + 52 = x^3 + 6x^2 + 11x + 6$$

$$x^3 + 52 = x^3 + 6x^2 + 11x + 6$$

$$x^3 + 6x^2 + 11x + 6 - x^3 - 52 = 0$$

$$6x^2 + 11x - 46 = 0$$

$$(6x + 23)(x - 2) = 0$$

$$6x + 23 = 0 \text{ or } x - 2 = 0$$

$$6x + 23 = 0 \text{ gives,}$$

$$6x = -23$$

$$x = -\frac{23}{6}, \text{ is impossible}$$

$$\text{So } x - 2 = 0 \text{ gives,}$$

$$x = 2$$

Substituting $x = 2$, volume of the cuboid

$$V = (2+1) \times (2+2) \times (2+3) = (3) \times (4) \times (5) = 60 \text{ cubic units.}$$

2. Construct a cubic equation with roots

(i) 1, 2, and 3 (ii) 1, 1, and −2

(iii) 2, $\frac{1}{2}$ and 1 [FRT - 2022]

Sol. (i) 1,2, and 3

Given roots are 1,2 and 3

Here $\alpha = 1, \beta = 2$ and $\gamma = 3$

A cubic polynomial equation whose roots are α, β, γ is

$x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma = 0$

$\Rightarrow x^3 - (1+2+3)x^2 + (2+6+3)x - 6 = 0$

$\Rightarrow x^3 - 6x^2 + 11x - 6 = 0.$

(ii) Here $\alpha = 1, \beta = 1$ and $\gamma = -2$

$\therefore$ The required cubic equation is

$x^3 - (1+1-2)x^2 + (1-2-2)x - (1)(1)(-2) = 0$

$x^3 - 0x^2 - 3x + 2 = 0$

$x^3 - 3x + 2 = 0.$

(iii) Here $\alpha = 2, \beta = \frac{1}{2}$ and $\gamma = 1$

$\therefore$ The cubic equation is

$x^3 - \left(2+\frac{1}{2}+1\right)x^2 + \left(1+\frac{1}{2}+2\right)x - 1 = 0$

$\Rightarrow 2x^3 - 7x^2 + 7x - 2 = 0$

3. If α, β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are

(i) $2\alpha, 2\beta, 2\gamma$, [Hy - 2019] **(ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$,**

(iii) $-\alpha, -\beta, -\gamma$

Sol. The roots of $x^3 + 2x^2 + 3x + 4 = 0$ are α, β, γ

$\therefore \alpha+\beta+\gamma = -$ co-efficient of $x^2 = -2$ ...(1)

$\alpha\beta+\beta\gamma+\gamma\alpha =$ co-efficient of $x = 3$ ...(2)

$-\alpha\beta\gamma = +4 \Rightarrow \alpha\beta\gamma = -4$ ...(3)

(i) Form a cubic equation whose roots are $2\alpha, 2\beta, 2\gamma$

$2\alpha+2\beta+2\gamma = 2(\alpha+\beta+\gamma) = 2(-2) = -4$ [from (1)]

$4\alpha\beta+4\beta\gamma+4\gamma\alpha = 4(\alpha\beta+\beta\gamma+\gamma\alpha) = 4(3) = 12$ [from (2)]

$(2\alpha)(2\beta)(2\gamma) = 8(\alpha\beta\gamma) = 8(-4) = -32$ [from (3)]

$\therefore$ The required cubic equation is

$x^3 - (2\alpha+2\beta+2\gamma)x^2 + (4\alpha\beta+4\beta\gamma+4\gamma\alpha)x - (2\alpha)(2\beta)(2\gamma) = 0$

$\Rightarrow x^3 - (-4)x^2 + 12x + 32 = 0$

$\Rightarrow x^3 + 4x^2 + 12x + 32 = 0$

(ii) Form the cubic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = \frac{3}{-4} = \frac{-3}{4}$

$\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha} = \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma} = \frac{-2}{-4} = \frac{1}{2}$

$\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma} = \frac{1}{-4} = -\frac{1}{4}$

$\therefore$ The required cubic equation is

$x^3 - \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)x^2 + \left(\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\right)x - \left(\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}\right) = 0$

$\Rightarrow x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{4} = 0$

Multiplying by 4 we get,

$4x^3 + 3x^2 + 2x + 1 = 0$

(iii) Form the equation whose roots are $\alpha - \beta - \gamma$

$\therefore -\alpha-\beta-\gamma = -(\alpha+\beta+\gamma) = -(-2) = 2$

$\alpha\beta+\beta\gamma+\gamma\alpha = 3$

$(-\alpha)(-\beta)(-\gamma) = -(\alpha\beta\gamma) = -(-4) = 4$

$\therefore$ The required cubic equation is

$x^3 - (-\alpha-\beta-\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - [(-\alpha)(-\beta)(-\gamma)] = 0$

$\Rightarrow x^3 - (2)x^2 + 3x - 4 = 0$

$\Rightarrow x^3 - 2x^2 + 3x - 4 = 0$

4. Solve the equation $3x^3 - 16x^2 + 23x - 6 = 0$ if the product of two roots is 1.

Sol. Given cubic equation is $3x^3 - 16x^2 + 23x - 6 = 0$

Let $\alpha, \frac{1}{\alpha}$ and γ be the roots of the equation [$\because$ product of two roots is 1]

$x^3 - \frac{16}{3}x^2 + \frac{23}{3}x - 2 = 0$ ...(1)

Comparing (1) with

$x^3 - \left(\frac{\alpha+1+\gamma}{\alpha}\right)x^2 + \left(\alpha\frac{1}{\alpha}+\frac{1}{\alpha}\cdot\gamma+\gamma\alpha\right)^x - \alpha\frac{1}{\alpha}\cdot\gamma = 0$ ...(2)

we get,

$\alpha+\frac{1}{\alpha}+\gamma = \frac{16}{3}$ ...(3)

$1+\frac{\gamma}{\alpha}+\gamma\alpha = \frac{23}{3}$

$\alpha\cdot\frac{1}{\alpha}\cdot\gamma = 2 \Rightarrow \gamma = 2$ ...(4)

Substituting $\gamma = 2$ in (3)

$$\alpha + \frac{1}{\alpha} + 2 = \frac{16}{3}$$

$$\Rightarrow \quad \alpha + \frac{1}{\alpha} = \frac{16}{3} - 2 = \frac{16-6}{3} = \frac{10}{3}$$

$$\Rightarrow \quad \frac{\alpha^2+1}{\alpha} = \frac{10}{3}$$

-10

9

$10 \qquad -1$

$\frac{10}{3} \qquad -\frac{1}{3}$

$(3\alpha + 10)\ (3\alpha - 1) = 0$

$$3\alpha^2 + 3 = 10\alpha$$

$$3\alpha^2 - 10\alpha + 3 = 0$$

$$(3\alpha + 10)\ (3\alpha - 1) = 0$$

$$\alpha = \frac{-10}{3} \text{ (or)}$$

$$\alpha = \frac{1}{3}$$

$$\alpha = \frac{-10}{3} \text{ is not possible } \Rightarrow \alpha = \frac{1}{3}$$

$$\left[\because \alpha = \frac{-10}{3} \text{ will not satisfy (5)}\right]$$

$\therefore$ The roots are 3, $\frac{1}{3}$, 2.

5. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$. [PTA - 2]

Sol. Given equation is $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Here $a = 2$, $b = -8$, $c = 6$, $d = 0$, $e = -3$

Let α, β, γ and δ be the roots of equation (1)

Then by Vieta's formula,

$$\Sigma_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a} = \frac{-(-8)}{2} = 4$$

$$\Sigma_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = \frac{6}{2} = 3$$

$$\Sigma_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a} = \frac{0}{2} = 0$$

$$\Sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a} = \frac{-3}{2}$$

Now, $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 4^2 - 2(3) = 16 - 6 = 10$$

6. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ if it is given that two of its roots are in the ratio 3:2.

Sol. Given $x^3 - 9x^2 + 14x + 24 = 0$

Two of its roots are in the ratio 3 : 2

$$\text{Let } \alpha : \beta = 3 : 2$$

$$\frac{\alpha}{\beta} = \frac{3}{2}$$

$$\therefore 2\alpha = 3\beta$$

Comparing with $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$$\Sigma_1 = (\alpha + \beta + \gamma) = 9$$

$$\Sigma_2 = (\alpha\beta + \alpha\gamma + \beta\gamma) = 14$$

$$\Sigma_3 = (\alpha\beta\gamma) = -24$$

$$\text{From } \alpha + \beta + \gamma = 9$$

$$2\alpha + 2\beta + 2\gamma = 18$$

Substituting $2\alpha = 3\beta$

$$3\beta + 2\beta + 2\gamma = 18$$

$$5\beta + 2\gamma = 18$$

$$2\gamma = 18 - 5\beta \quad \ldots (1)$$

$$\text{From } \alpha\beta\gamma = -24$$

$$2\alpha\beta\gamma = -48$$

Substituting $2\alpha = 3\beta$

$$(3\beta)\beta\gamma = -48$$

$$3\beta^2\gamma = -48$$

$$\beta^2\gamma = -16$$

Multiplying by 2, $\beta^2 2\gamma = -32$

Substituting $2\gamma = 18 - 5\beta$

$$\beta^2(18 - 5\beta) = -32$$

$$18\beta^2 - 5\beta^3 = -32$$

$$5\beta^3 - 18\beta^2 - 32 = 0$$

$$(\beta - 4)(5\beta^2 + 2\beta + 8) = 0$$

$\beta - 4 = 0$, gives $\beta = 4$

Substituting $\beta = 4$ in $2\alpha = 3\beta$

$$2\alpha = 3(4)$$

$$2\alpha = 12$$

$$\therefore \alpha = 6$$

Substituting $\beta = 4$ in $2\gamma = 18 - 5\beta$

$$2\gamma = 18 - 5(4)$$

$$2\gamma = 18 - 20$$

$$2\gamma = -2$$

$$\therefore \gamma = -1$$

Hence the roots are 6, 4 and -1

7. If α, β and γ are the roots of the polynomial equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\sum \frac{\alpha}{\beta\gamma}$ in terms of the coefficients.

Sol. Given α, β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$

$$\therefore \alpha+\beta+\gamma = \frac{-b}{a}$$
$$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$$
$$\alpha\beta\gamma = \frac{-d}{a}$$
$$\text{Now, } \sum\frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta}$$
$$= \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$$
$$= \frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}$$
$$= \frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{-\frac{d}{a}}$$
$$= \frac{\frac{b^2}{a^2}-\frac{2c}{a}}{-\frac{d}{a}} \Rightarrow \frac{b^2-2ac}{a^2}\times\frac{-a}{d}$$
$$\therefore \frac{\sum\alpha}{\beta\gamma} = -\frac{(b^2-2ac)}{ad} = \frac{2ac-b^2}{ad}$$

8. If α, β, γ and δ are the roots of the polynomial equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$, find a quadratic equation with integer coefficients whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$

Sol. Given polynomial equation is **[Sep - 2020]**

$$2x^4+5x^3-7x^2+8 = 0$$

Here $a = 2, b = 5, c = -7, d = 0, e = 8$

By Vieta's formula,

$$\alpha+\beta+\gamma+\delta = \frac{-b}{a} = \frac{-5}{2}$$
$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}=\frac{-7}{2}$$
$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = \frac{-d}{a}=0$$
$$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{8}{2}=4$$

Given roots of the quadratic equation are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$

$$\therefore \text{sum of the roots} = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$$
$$= \left(\frac{-5}{2}+4\right) = \frac{-5+8}{2} = \frac{3}{2}$$

$$\text{Product of the roots} = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$$
$$= \left(\frac{-5}{2}\right)(4) = \frac{-20}{2} = -10$$

$\therefore$ The required quadratic equation is $x^2 - x$ (sum of the roots) + product of the roots = 0

$$\Rightarrow x^2 - x\left(\frac{3}{2}\right)-10 = 0$$
$$\Rightarrow 2x^2-3x-20 = 0$$

9. If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}} = 0$.

Sol. Given p, q are the roots of $lx^2 + nx + n = 0$

$$p+q = -\frac{n}{l} \text{ and } pq = \frac{n}{l}$$

$$\text{Consider } \frac{(p+q)^2}{pq} = \frac{\left(\frac{-n}{l}\right)^2}{\left(\frac{n}{l}\right)} = \frac{n^2}{l^2}\times\frac{l}{n} = \frac{n}{l}$$

Taking square root on both sides

$$\sqrt{\frac{(p+q)^2}{pq}} = \sqrt{\frac{n}{l}}$$
$$\Rightarrow \pm\frac{(p+q)}{\sqrt{pq}} = \sqrt{\frac{n}{l}}$$
$$\text{Consider } -\frac{(p+q)}{\sqrt{pq}} = \sqrt{\frac{n}{l}}$$
$$\Rightarrow \frac{(p+q)}{\sqrt{pq}} = -\sqrt{\frac{n}{l}}$$
$$\frac{p}{\sqrt{pq}}+\frac{q}{\sqrt{pq}} = -\sqrt{\frac{n}{l}}$$
$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}} = 0$$

Hence proved.

10. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.

Sol. Given equation are $x^2+px+q = 0$ ...(1)

and $x^2+p'x+q' = 0$ ...(2)

Let α be the common root for (1) and (2)

$\therefore \alpha^2+p\alpha+q = 0$ ...(3)

and $\alpha^2+p'\alpha+q' = 0$ ...(4)

CHAPTER 4

INVERSE TRIGONOMETRIC FUNCTIONS

MUST KNOW DEFINITIONS

- f is **periodic** if there exists $p > 0$ such that for all x in the domain of f, $x + p$ is in the domain of f and $f(x + p) = f(x)$.
- The smallest of all such number is called the **period** of the function f.
- A real values function f is an **even** function if for all x in the domain of f, $-x$ is also in the domain of f and $f(-x) = f(x)$.
- A real values function f is an **odd** function if for all x in the domain of f, $-x$ is also in the domain of f and $f(-x) = -f(x)$.
- **Amplitude** of a function is the height from the x-axis to its maximum or minimum.
- The period is the distance required for the function to complete one full cycle.
- The inverse sine function $\sin^{-1}: [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is defined by $\sin^{-1}(x) = y$ if and only if $\sin y = x$ and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- The inverse tangent function $\tan^{-1}$ $(-\infty, \infty) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is defined by $\tan^{-1}(x) = y$ if and only if $\tan y = x$ and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- The inverse cosecant function $\operatorname{cosec}^{-1}$: $(-\infty, -1] \cup [1, \infty)$ $\left[\frac{-\pi}{2}, 0\right) \cup \left[0, \frac{\pi}{2}\right)$ is defined by $\operatorname{cosec}^{-1}(x) = y$ if and only if $\operatorname{cosec} y = x$ and $y \in \left[\frac{-\pi}{2}, 0\right) \cup \left[0, \frac{\pi}{2}\right)$
- The inverse secant function $\sec^{-1}$: $R\backslash(-1, 1) \to [0, \pi]/\left\{\frac{\pi}{2}\right\}$ is defined by $\sec^{-1}(x) = y$ whenever $\sec y = x$ and $y \in [0, \pi]/\left\{\frac{\pi}{2}\right\}$
- The inverse cotangent function $\cot^{-1}: (-\infty, \infty) \to [0, \pi]$ is defined by $\cot^{-1}(x) = y$ if and only if $\cot y = x$ and $y \in [0, \pi]$.

IMPORTANT FORMULAE TO REMEMBER

Properties of Inverse Trigonometric Functions

✦ **Property-I**

(i) $\sin^{-1}(\sin\theta) = \theta$, if $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, if $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, if $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, if $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

(v) $\sec^{-1}(\sec\theta) = \theta$, if $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

(vi) $\cot^{-1}(\cot\theta) = \theta$, if $\theta \in [0, \pi]$

✦ **Property-II**

(i) $\sin(\sin^{-1}x) = x$, if $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x$, if $x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x$, if $x \in \mathbb{R}$

(iv) $\text{cosec}\,(\text{cosec}^{-1}x) = x$, if $x \in \mathbb{R}\setminus(-1, 1)$

(v) $\sec(\sec^{-1}x) = x$, if $x \in \mathbb{R}\setminus(-1, 1)$

(vi) $\cot(\cot^{-1}x) = x$, if $x \in \mathbb{R}$

✦ **Property-III (Reciprocal inverse identities)**

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}\,x$, if $x \in \mathbb{R} \setminus(-1, 1)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec x$, if $x \in \mathbb{R} \setminus(-1, 1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & \text{if } x > 0 \\ -\pi + \cot^{-1}x & \text{if } x < 0 \end{cases}$

Property-IV (Reflection identities)

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, if $x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$, if $x \in \mathbb{R}$

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R}\setminus(-1, 1)$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, if $x \in [-1, 1]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, if $x \geq 1$ or $x \in \mathbb{R}\setminus(-1, 1)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, if $x \in \mathbb{R}$

Property-V (co-function inverse identities)

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $x \in [-1, 1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in \mathbb{R}$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$, $x \in \mathbb{R}\setminus(-1, 1)$ or $|x| \geq 1$

Property-VI

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, where either $x^2 + y^2 \leq 1$ or $xy < 0$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, where either $x^2 + y^2 \leq 1$ or $xy > 0$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x + y \geq 0$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x \neq y$

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, if $xy > -1$

Property-VII

(i) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $|x| < 1$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \geq 0$

(iii) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $|x| \leq 1$,

Property-VIII

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$, if $|x| \leq \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$, if $\frac{1}{\sqrt{2}} \leq x \leq 1$

Property-IX

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$

(ii) $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$

(iii) $\sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-1 < x < 1$

(iv) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$

(v) $\cos^{-1}x = A - \sin^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$

(vi) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, if $x > 0$

Property-X

(i) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(ii) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$

- The period of $f = g \pm h$ is l cm {period of g, period of h}, whenever they exist.
- The graph of an even function is symmetric with respect to origin and the graph of an even function is symmetric about y-axis.

EXERCISE 4.1

1. Find all the values of x such that
(i) $-10\pi \leq x \leq 10\pi$ and $\sin x = 0$
(ii) $-3\pi \leq x \leq 3\pi$ and $\sin x = -1$.

Sol.

(i) Given $\sin x = 0$

$\Rightarrow \sin x = \sin 0 \qquad \Rightarrow x = n\pi,\ n \in z.$

Since $-10\pi \leq x \leq 10\pi$, n can take the values only from -10 to $+10$.

$\therefore x = n\pi$, where $n = 0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm7, \pm8, \pm9, \pm10$

(ii) $\sin x = -1$

$\Rightarrow \sin x = \sin\left(\frac{-\pi}{2}\right) \Rightarrow x = \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$ .....,

$\Rightarrow x = (4n-1)\frac{\pi}{2}$, $n \in$ Z. $\Rightarrow x = (4n-1)\frac{\pi}{2}$,

n takes the values 0, ±1,

2. Find the period and amplitude of

(i) $y = \sin 7x$ (ii) $y = -\sin\left(\frac{1}{3}x\right)$
(iii) $y = 4\sin(-2x)$.

Sol.

(i) Given $y = \sin 7x$

The amplitude of $\sin x$ is 1 [Max of sin curve is 1]

$\Rightarrow$ amplitude of $\sin 7x$ is also 1.

If p is the period of the function, then $f(x+p) = f(x)$.

Since the period of sine function is 2π. The period of $\sin 7x$ is $\frac{2\pi}{7}$ since $\sin 7\left(\frac{2\pi}{7}\right) = \sin 2\pi$.

(ii) The amplitude $\sin x$ is 1

$\Rightarrow$ amplitude of $-\sin\left(\frac{1}{3}x\right)$ is also 1.

The period of $-\sin\left(\frac{1}{3}x\right)$ is $\frac{1}{3}x = 2\pi \Rightarrow x = 6\pi$.

(iii) The amplitude of $\sin x$ is 1

$\Rightarrow$ amplitude of $\sin(-2x)$ is 1.

$\therefore$ Amplitude $4\sin(-2x)$ is $4 \times 1 = 4$.

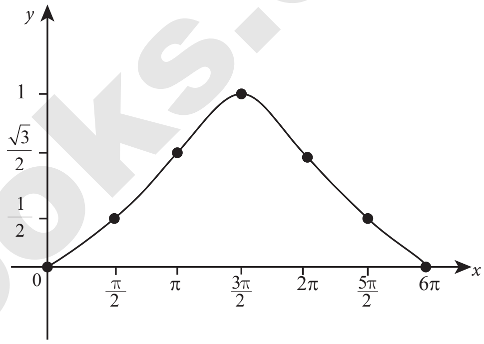
The period of $\sin(-2x)$ is $2x = 2\pi \Rightarrow x = \frac{2\pi}{2} = \pi$

3. Sketch the graph of $y = \sin\left(\frac{1}{3}x\right)$ for $0 \leq x \leq 6\pi$.

Sol.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	6π
$\sin\left(\frac{1}{3}x\right)$	0	$\sin\left(\frac{\pi}{6}\right)$	$\sin\left(\frac{\pi}{3}\right)$	$\sin\left(\frac{1}{3}\times\frac{3\pi}{2}\right)$	$\sin\left(\frac{1}{3}\times 2\pi\right)$	$\sin\left(\frac{1}{3}\times\frac{5\pi}{2}\right)$	$\sin 2\pi$
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

Plot the points $(0, 0)$ $\left(\frac{\pi}{2}, \frac{1}{2}\right)$, $\left(\pi, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{3\pi}{2}, 1\right)$, $\left(2\pi, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{5\pi}{2}, \frac{1}{2}\right)$ and $(6\pi, 0)$



4. Find the value of

(i) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

(ii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$ **[PTA - 3; March - 2020]**

Sol.

(i) $= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$ $\left[\because \frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right.$ which is the principal domain of sine function]

$= \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$ $[\because \sin(\pi - \theta) = \sin\theta]$

$= \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $= \sin^{-1}\left(\sin\left(\pi + \frac{\pi}{4}\right)\right)$ $\because \frac{5\pi}{4} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

5. For what value of x does $\sin x = \sin^{-1}x$?

Sol. **[Aug. - 2021]**

$$\begin{aligned} \text{Let } y &= \sin^{-1}x \\ \text{When } y &= 0,\ 0 = \sin^{-1}x \\ \Rightarrow \quad \sin(0) &= \sin(\sin^{-1}(x)) \\ \Rightarrow \quad \sin 0 &= x \\ \Rightarrow \quad x &= 0 \end{aligned}$$

$\therefore x = 0$, $\sin x = \sin^{-1}x$

CHAPTER 12

DISCRETE MATHEMATICS

MUST KNOW DEFINITIONS

- A **binary operation** * on S is defined as follows: $\forall\ a, b \in$ S, $a * b$ is unique and $a * b \in$ S
- A binary operation * defined by *: S × S → S; $(a, b) = a * \in$ S must always lie in the given set and not in the complement of it. Then S is closed with respect to *.
- A binary operation * defined on a non empty set S is said to satisfy the **commutative property** if $a * b = b * a\ \forall\ a, b \in$ S
- If $a * (b * c) = (a * b) * c\ \forall\ a, b \in$ S, then S is said to satisfy the **associative property**.
- An element $e \in$ S is said to satisfy the **identity element** of s if $\forall\ a \in$ S, $a * e, = e * a = a$.
- If for every $a \in$ S, there exists b in S such that $a * b = b * a = e$ then $b \in$ S is said to be the inverse element of a.
- In an algebraic structure, the identity element and the inverse of an element must be unique.
- A Boolean matrix is a real matrix whose entries are either 0 or 1.
- Joint of A and B, $A \vee B = [a_{ij}] \vee [b_{ij}] = [a_{ij} \vee b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1 & \text{if either } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{if both } a_{ij} = 0,\ b_{ij} = 0 \end{cases}$
- Meet of A and B, $A \wedge B = [a_{ij} \wedge b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1 & \text{if both } a_{ij} = 1, b_{ij} = 1 \\ 0 & \text{if either } a_{ij} = 0, b_{ij} = 0 \end{cases}$
- **Addition moduls** n

  Let $a, b \in \mathbb{Z}n$. Then $a +_n b$ = the remainder of $a + b$ on division by n.
- Multiplication moduls n

  Let $a, b \in \mathbb{Z}n$. Then $a \times_n b$ = the remainder of $a \times b$ on division by n.
- A statement is said to be a tautology (II) if its truth value is always T irrespective of the truth values of its compound statements.
- A statement is a **contradiction** ($\mathbb{F}$) if its truth value is always F irrespective of the truth value of its compound statements.
- A dual is obtained by replacing II by $\mathbb{F}$ and $\mathbb{F}$ by II

IMPORTANT FORMULAE TO REMEMBER

✦ Truth table for negation ~

p	$\sim p$
T	F
F	T

✦ Truth table for AND ($\wedge$) conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

✦ Truth table for OR($\vee$) Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

✦ Truth table for $p \rightarrow q$ (conditional statement)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

✦ Truth table for $p \leftrightarrow q$ (bi conditional statement)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

✦ Truth table for $p \,\bar{\vee}\, q$

p	q	$p \,\bar{\vee}\, q$
T	T	F
T	F	T
F	T	T
F	F	F

✦ Any two compound statements A and B are said to be logically equivalent if the columns corresponding to A and B in the truth table have identical truth values. (A ≡ B) or A ⇔ B.

✦ Laws of equivalence.

1. Idempotent laws: (i) $p \vee q \equiv p$ (ii) $p \wedge q \equiv p$
2. Commutative laws: (i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$
3. Associative laws: (i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
4. Distributive laws: (i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. Identity laws: (i) $p \vee \mathrm{T} = \mathrm{T}$ and $p \vee \mathbb{F} \equiv p$ (ii) $p \wedge \mathrm{T} = p$ and $p \wedge \mathbb{F} \equiv p$
6. Component laws: (i) $p \vee \sim p = \mathrm{T}$ and $p \wedge \sim p \equiv \mathbb{F}$ (ii) $\sim \mathrm{T} \equiv \mathbb{F}$ and $\sim \mathbb{F} \equiv \mathrm{T}$
7. Involution law : (Double negation law) $\sim(\sim p) \equiv p$
8. DeMorgan's law: (i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$ (ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$
9. Absorption laws: (i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

EXERCISE 12.1

1. Determine whether * is a binary operation on the sets given below.

(i) $a*b = a.|b|$ on $\mathbb{R}$.

(ii) $a*b = \min(a, b)$ on A ={1,2,3,4,5}

(iii) $(a*b) = a\sqrt{b}$ is binary on $\mathbb{R}$.

Sol. (i) **$a*b = a\,.\,|b|$ on $\mathbb{R}$.**

Given $a * b = a \cdot |b|$ on $\mathbb{R}$.

Let $a, b \in \mathbb{R}$.

Then $a * b = a\,.\,|b| \in \mathbb{R}$.

Since $a\,.\,|b| = ab$ if $b > 0 = -ab$ if $b < 0$

$\therefore a * (b) = a\,.\,|b| \in \mathbb{R}$

So,* is a binary operation on $\mathbb{R}$.

(ii) **$a*b = \min(a, b)$ on A = {1, 2, 3, 4, 5}**

Let $a, b \in$ A

Then $\min(a, b) = a$ or b and $a, b \in$ A

$\therefore a * b = \min(a, b) \in$ A

So,* is a binary operation on A.

(iii) **$(a*b) = a\sqrt{b}$ is binary on $\mathbb{R}$.**

Let $a, b \in \mathbb{R}$

[$\therefore$ Square root of negative numbers does not belong to $\mathbb{R}$]

$a * b = a\sqrt{b} \notin \mathbb{R}$ if $b < 0$

So,* is not a binary operation on $\mathbb{R}$.

2. On $\mathbb{Z}$, define * by $(m * n) = m^n + n^m : \forall m, n \in \mathbb{Z}$. Is * binary on $\mathbb{Z}$? [PTA -3]

Sol. Given $m * n = m^n + n^m \;\forall\, m, n \in \mathbb{Z}$

Let $m, n \in \mathbb{Z}$

Consider $m = -3, n = 2$

$$\therefore m * n = (-3)^2 + 2^{-3} = 9 + \frac{1}{8} = \frac{72+1}{8} = \frac{73}{8} \notin \mathbb{Z}$$

$\therefore$ * is not a binary operation on $\mathbb{Z}$.

3. Let * be defined on $\mathbb{R}$ by $(a*b) = a + b + ab - 7$. Is * binary on $\mathbb{R}$? If so, find $3 * \left(\frac{-7}{15}\right)$. [PTA - 2] [May - 2022]

Sol. Given $a * b = a + b + ab - 7$

Let $a, b \in \mathbb{R}$

$a * b = a + b + ab - 7 \in \mathbb{R}$,

$\therefore$ * is binary operation on $\mathbb{R}$.

$$3 * \left(\frac{-7}{15}\right) = 3 - \frac{7}{15} + \cancel{3}^{1}\left(\frac{-7}{\cancel{15}_{5}}\right) - 7$$

[Here $a = 3, b = \frac{-7}{15}$]

$$= 3 - \frac{7}{15} - \frac{7}{5} - 7$$

$$= \frac{45 - 7 - 21 - 105}{15} = \frac{-88}{15}$$

$$\therefore 3 * \left(\frac{-7}{15}\right) = \frac{-88}{15}$$

4. Let A = $\{a + \sqrt{5}b : a, b \in \mathbb{Z}\}$. Check whether the usual multiplication is a binary operation on A. [PTA - 5]

Sol. Given A = $\{a + \sqrt{5}\,b, a, b \in z\}$

Let C = $a + \sqrt{5}\,b$,

B = $c + \sqrt{5}\,d \in$ A

where $a, b, c, d \in \mathbb{Z}$

[$\because ac + 5\,bd \in \mathbb{Z}$ and $ad + bc \in \mathbb{Z}$]

$\therefore$ B = $(a + \sqrt{5}\,b)\,.\,(c + \sqrt{5}\,d)$

$= ac + \sqrt{5}\,ad + cb\sqrt{5} + 5\,bd$

$= (ac + 5\,bd) + \sqrt{5}\,(ad + bc) \in$ A

$\therefore$ C . B $\in$ A $\forall\, a, b, c, d, \in \mathbb{Z}$

$\therefore$ Usual multiplication is a binary operation on A.

5. (i) Define an operation*on $\mathbb{Q}$ as follows : $a*b = \left(\frac{a+b}{2}\right)$; $a, b \in \mathbb{Q}$. Examine the closure, commutative, and associative properties satisfied by * on $\mathbb{Q}$.

(ii) Define an operation*on $\mathbb{Q}$ as follows: $a*b = \left(\frac{a+b}{2}\right)$, $a, b \in \mathbb{Q}$. Examine the existence of identity and the existence of inverse for the operation * on $\mathbb{Q}$. [Aug. - 2021]

Sol. (i) Given $a * b = \dfrac{a+b}{2} \;\forall a, b \in \mathbb{Q}$.

(i) **Closure property:**

Let $a, b \in \mathbb{Q}$

$$\therefore a * b = \frac{a+b}{2} \in \mathbb{Q}$$

[$\because$ Addition and division are closed on $\mathbb{Q}$.]

* is closed on $\mathbb{Q}$.

(ii) **Commutative property :**

Let $a, b \in \mathbb{Q}$

$$\text{Then } a * b = \frac{a+b}{2} = \frac{b+a}{2} = b * a$$

$\therefore a * b = b * a \quad \forall a, b \in \mathbb{Q}$

$\therefore$* is commutative on $\mathbb{Q}$.

(iii) **Associative property :**

Let $a, b, c \in \mathbb{Q}$

$a * (b * c) = (a * b) * c$

Let $a = 2, b = 3, c = -5$

$$\therefore a * (b * c) = 2 * (3 * -5) = 2 * \left(\frac{3-5}{2}\right)$$

$$= 2 * (-1) = \frac{2 + (-1)}{2} = \frac{1}{2} \quad ...(1)$$

Now $(a * b) * c = (2 * 3) * (-5)$

$$= \left(\frac{2+3}{2}\right) * (-5) = \frac{5}{2} * -5 = \frac{\frac{5}{2}+(-5)}{2}$$

$$= \frac{5-10}{4} = \frac{-5}{4} \quad ...(2)$$

From (1) & (2), $a * (b * c) \neq (a * b) * c$

$\therefore$ * is not associative on $\mathbb{Q}$.

(ii) Given $a * b = \frac{a+b}{2}$, where $a, b \in \mathbb{Q}$

Let $a, b \in \mathbb{Q}$

An element c has to found out such that $a * e = e * a = a$.

Let $a = 5$, Then $5 * e = 5$

$\Rightarrow \frac{5+e}{2} = 5 \Rightarrow 5 + e = 10$

Let $a = \frac{2}{3}$. Then $\frac{2}{3} * e = \frac{2}{3}$

$\Rightarrow \frac{\frac{2}{3}+e}{2} = \frac{2}{3} \Rightarrow \frac{2}{3} + e = \frac{4}{3}$

$\Rightarrow e = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

Since identity differs for every element, the identity does not exist for Q.

$\therefore$ * has no identity on Q.

$\therefore$ * has no inverse on Q.

Hence, identity and inverse does not exist for Q under the given binary operation *.

6. Fill in the following table so that the binary operation * on A ={*a*,*b*,*c*} is commutative.

*	*a*	*b*	*c*
a	*b*		
b	*c*	*b*	*a*
c	*a*		*c*

Sol. Given * on A is commutative

Given $b * a = c \Rightarrow a * b = c$

Given $c * a = a \Rightarrow a * c = a$

Given $b * c = a \Rightarrow c * b = a$

Hence

*	*a*	*b*	*c*
a	*b*	*c*	*a*
b	*c*	*b*	*a*
c	*a*	*a*	*c*

7. Consider the binary operation * defined on the set A ={*a*,*b*,*c*,*d*} by the following table:

*	*a*	*b*	*c*	*d*
a	*a*	*c*	*b*	*d*
b	*d*	*a*	*b*	*c*
c	*c*	*d*	*a*	*a*
d	*d*	*b*	*a*	*c*

Is it commutative and associative?

Sol. Given A = {*a*, *b*, *c*, *d*} and * is defined as follows.

*	*a*	*b*	*c*	*d*
a	*a*	*c*	*b*	*d*
b	*d*	*a*	*b*	*c*
c	*c*	*d*	*a*	*a*
d	*d*	*b*	*a*	*c*

From the table,

(i) $a * b = c$ and $b * a = d$

$\Rightarrow$ * is not commutative on A.

(ii) Let us verify $a * (b * c) = (a * b) * c$

$\Rightarrow a * (b) = c * c$

$\Rightarrow c \neq a$

No. The given operation is not commutative and associative.

8. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ **be any three boolean matrices of the same type. Find (i) A$\vee$B (ii) A$\wedge$B (iii) (A$\vee$B)$\wedge$C (iv) (A$\wedge$B)$\vee$C .**

Sol. Given $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

and $C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

(i) $A \vee B$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\vee 0 & 0\vee 1 & 1\vee 0 & 0\vee 1 \\ 0\vee 1 & 1\vee 0 & 0\vee 1 & 1\vee 0 \\ 1\vee 1 & 0\vee 0 & 0\vee 0 & 1\vee 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$[\because a \vee b = \max(a, b)]$

(ii) $A \wedge B$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\wedge 0 & 0\wedge 1 & 1\wedge 0 & 0\wedge 1 \\ 0\wedge 1 & 1\wedge 0 & 0\wedge 1 & 1\wedge 0 \\ 1\wedge 1 & 0\wedge 0 & 0\wedge 0 & 1\wedge 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$[\because a \wedge b = \min(a, b)]$

(iii) $(A \vee B) \wedge C$

From (1),

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(iv) $(A \wedge B) \vee C$

From (ii)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$(A \wedge B) \vee C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

9. (i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\} \right\}$ and let * be the matrix multiplication. Determine whether M is closed under*. If so, examine the commutative and associative properties satisfied by * on M .

(ii) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\} \right\}$ and let * be the matrix multiplication. Determine whether M is closed under *. If so, examine the existence of identity, existence of inverse properties for the operation * on M .

Sol. (i) Given $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\} \right\}$ and * be the matrix multiplication.

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M$

where $x, y \in \mathbb{R} - \{0\}$.

$$A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$$

$[\because 2xy \in \mathbb{R} - \{0\}]$

$\therefore$ M is closed under *.

Commutative property:

We know $A * B = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix}$ ...(1)

Let $x, y \in \mathbb{R} - \{0\}$

$$\text{Now } B * A = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix}$$

From (1) & (2), $A * B = B * A$

$\therefore$ * has commutative property on M.

Associative property:

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$,

$B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$ and

$C = \begin{pmatrix} z & z \\ z & z \end{pmatrix}$

for $x, y, z \in \mathbb{R} - \{0\}$

$$(A * B) * C = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} * \begin{pmatrix} z & z \\ z & z \end{pmatrix}$$

$$= \begin{pmatrix} 2xyz+2xyz & 2xyz+2xyz \\ 2xyz+2xyz & 2xyz+2xyz \end{pmatrix}$$

$$= \begin{pmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{pmatrix} \quad ...(1)$$

$$\text{Now } A * (B * C) = A * \begin{pmatrix} 2yz & 2yz \\ 2yz & 2yz \end{pmatrix}$$

$$= \begin{pmatrix} x & x \\ x & x \end{pmatrix} * \begin{pmatrix} 2yz & 2yz \\ 2yz & 2yz \end{pmatrix}$$

$$= \begin{pmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{pmatrix} \quad ...(2)$$

From (1) & (2), $(A * B) * C = A * (B * C)$

$\therefore$ * has associative property on M.

(ii) **(1) Closure**

Let A $= \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and

B $= \begin{pmatrix} y & y \\ y & y \end{pmatrix} : x, y \in \mathbb{R} - (0).$

Now, AB $= \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$

$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in$ M.

Since, $x, y \in \mathbb{R} - (0)$ gives xy also $y \in \mathbb{R} - (0)$

So, AB $y \in$ M $\Rightarrow$ A * B $\in$ M

$\therefore$ * is closed on M.

(2) Existence of Identity :

Let A $= \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and

E $= \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be the identity,

such that : $a, e \in \mathbb{R} - (0)$.

Hence M = (A, E)

Now, A * E = E * A = A

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2xe = x$$

$$2e = 1$$

$$e = \frac{1}{2} \in \mathbb{R} - (0)$$

$\therefore$ E $= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the identity $\in$ M

$\therefore$ * has identity on M.

(3) Existence of Inverse :

Let A $= \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and

$A^{-1} = \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix}$

be the inverse of A.

Then * $\quad A * A^{-1} = A^{-1} * A = E$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix} = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$$

$$\begin{pmatrix} 2xx^{-1} & 2xx^{-1} \\ 2xx^{-1} & 2xx^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$2xx^{-1} = \frac{1}{2}$$

$$x^{-1} = \frac{1}{4x}, \in \mathbb{R} - (0)$$

$$e = \frac{1}{2} \in \mathbb{R} - (0)$$

$\therefore A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix}$ is the inverse of A $\in$ M

$\therefore$ * has inverse on M.

10. (i) Let A be $\mathbb{Q} \setminus \{1\}$. Define * on A by $x*y = x + y - xy$. Is * binary on A ? If so, examine the commutative and associative properties satisfied by * on A .

(ii) Let A be $\mathbb{Q} \setminus \{1\}$. Define * on A by $x*y = x + y - xy$. Is * binary on A ? If so, examine the existence of identity, existence of inverse properties for the operation * on A .

Sol. (i) Given A = {Q \ {1}}

A is defined on A by $x * y = x + y - xy$.

Let $x, y \neq 1$

$\therefore x * y = x + y - xy$

Now to prove that $x + y - xy \neq 1$

Let us assume that $x + y - xy = 1$

$$x + y - xy - 1 = 0$$

$$(x-1) - y(x-1) = 0$$

$$(x-1)(1-y) = 0$$

$x = 1$ or $y = 1$ which is a false $\quad [\because x, y \neq 1]$

$\therefore$ Our assumption is wrong.

$\therefore x + y - xy \neq 1$

* is a binary operation on A.

Commutative property:

Let $x, y \in A \Rightarrow x, y \neq 1$

$\therefore x * y = x + y - xy$

and $y * x = y + x - yx$

$\Rightarrow x * y = y * x \ \forall x, y \in A$

A has commutative property under *.

12th STD

INSTANT SUPPLEMENTARY EXAM - JULY 2022

Reg. No.

Part - III

Mathematics (with answers)

TIME ALLOWED : 3.00 Hours] [MAXIMUM MARKS : 90

Instructions :

(1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams

PART - I

Note : (i) **All** questions are **compulsory.**

(ii) Choose the most appropriate answer from the given **four** alternatives and write the option code and the corresponding answer.

20 × 1 = 20

1. If $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is :

(a) $\frac{-4}{5}$ (b) $\frac{-3}{5}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

2. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$ then $(A^T)^{-1} =$

(a) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

3. If $z = x + iy$ is a complex number such that $|z+2| = |z-2|$, then the locus of z is ______.

(a) real axis (b) imaginary axis

(c) ellipse (d) circle

4. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is :

(a) 0 (b) 1 (c) -1 (d) i

5. A zero of $x^3 + 64$ is :

(a) 0 (b) 4 (c) $4i$ (d) -4

6. The principal value of $\cos^{-1}\left(\cos\frac{\pi}{6}\right)$:

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{-\pi}{6}$ (d) $\frac{\pi}{3}$

7. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ having centre at (0,3) is :

(a) $x^2 + y^2 - 6y - 7 = 0$ (b) $x^2 + y^2 - 6y + 7 = 0$

(c) $x^2 + y^2 - 6y - 5 = 0$ (d) $x^2 + y^2 - 6y + 5 = 0$

8. The eccentricity of the hyperbola $\frac{x^2}{16} - \frac{(y-3)^2}{4} = 1$ is :

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\sqrt{5}$ (d) $\frac{1}{2}$

9. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$ then :

(a) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 1$ (b) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = -1$

(c) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$ (d) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 2$

10. Distance from the origin to the plane : $3x - 6y + 2z + 7 = 0$ is :

(a) 0 (b) 1 (c) 2 (d) 3

11. A stone is thrown up vertically. The height it reaches at time t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in time t seconds is given by :

(a) 2 (b) 2.5 (c) 3 (d) 3.5

12. The angle between the parabola $y^2 = x$ and $x^2 = y$ at the origin is :

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) 0

13. The percentage error of fifth root of 31 is approximately how many times the percentage error in 31?

(a) $\frac{1}{31}$ (b) $\frac{1}{5}$ (c) 5 (d) 31

14. The value of $\int_{-1}^{2} |x|dx$ is :

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{7}{2}$

15. The area between $y^2 = 4x$ and its latus rectum is :

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) $\frac{8}{3}$ (d) $\frac{5}{3}$

16. The order of the differential equation of all circles with centre at (h, k) and radius 'a' is ____. (where h, k are arbitrary constants)

(a) 2 (b) 3 (c) 4 (d) 1

17. The differential equation representing the family of curves $y = A\cos(x + B)$, where A and B are parameters, is :

(a) $\frac{d^2y}{dx^2} - y = 0$ (b) $\frac{d^2y}{dx^2} + y = 0$

(c) $\frac{d^2y}{dx^2} = 0$ (d) $\frac{d^2x}{dy^2} = 0$

18. If a fair die is thrown once then the probability to get a prime number on the face is :

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

19. A random variable X takes the probability mass function :

X	–2	3	1
P(X = x)	$\frac{\lambda}{6}$	$\frac{\lambda}{4}$	$\frac{\lambda}{12}$

The value of λ is :

(a) 1 (b) 2 (c) 3 (d) 4

20. Which one of the following is a binary operation on N ?

(a) Subtraction (b) Multiplication
(c) Division (d) All of the above

PART - II

Note : (i) Answer **any seven** questions.
(ii) Question number **30** is **compulsory.**

7 × 2 = 14

21. Find df for $f(x) = x^2 + 3x$ and evaluate it for $x = 3$ and $dx = 0.02$.

22. If α and β are the roots of $x^2 + 5x + 6 = 0$, then show that $\alpha^2 + \beta^2 = 13$.

23. Find the value of $\sin^{-1}(1) + \cos^{-1}(1)$.

24. Find the acute angle between the two straight lines. $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$

25. Find the tangent to the curve $y = x^2 - x^4$ at (1, 0).

26. If $z_1 = 3$, $z_2 = -7i$ and $z_3 = 5 + 4i$, show that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

27. Show that $y = ae^x + be^{-x}$ is a solution of the differential equation $y'' - y = 0$.

28. A random variable X has the following probability mass function.

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Show that the value of k is $\frac{1}{6}$.

29. Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function X is : $f(x) = \begin{cases} k, & 200 \le x \le 600 \\ 0 & \text{otherwise} \end{cases}$

Find the value of k

30. Form the differential equation of the curve $y = ax^2 + bx + c$ where a, b and c are arbitrary constants.

PART - III

Note : (i) Answer **any seven** questions.
(ii) Question number **40** is **compulsory.**

7 × 3 = 21

31. Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$.

32. Find the rank of the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$.

33. Show that the square roots of $6 - 8i$ are $\pm(2\sqrt{2} - i\sqrt{2})$.

34. Prove that the roots of the equation $x^4 - 3x^2 - 4 = 0$ are $\pm 2, \pm i$.

35. Find centre and radius of the circle $x^2 + y^2 + 6x - 4y + 4 = 0$

36. A particle acted on by constant forces $8\hat{i} + 2\hat{j} - 6\hat{k}$ and $6\hat{i} + 2\hat{j} - 2\hat{k}$ is displaced from the point (1, 2, 3) to the point (5, 4, 1). Find the total work done by the forces.

37. Show that $\lim\limits_{x \to 0^+}$, $x \log x$ is 0.

38. A circular plate expands uniformly under the influence of heat. If its radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area.

39. Verify (i) Closure property (ii) Commutative property of the following operation on the given set $(a * b) = a^b$, $\forall\ a, b$ N (exponentiation property).

40. Prove that $\int_0^1 xe^x\,dx = 1$.

PART - IV

Note : Answer **all** the following questions. **7 × 5 = 35**

41. (a) Solve the system of linear equations by Cramer's Rule $3x + 3y - z = 11$, $2x - y + 2z = 9$. $4x + 3y + 2z = 25$.

OR

(b) A particle is fired straight up from the ground to reach a height of s feet in t seconds, where $s(t) = 128t - 16t^2$.

(i) Compute the maximum height of the particle reached.

(ii) What is the velocity when the particle hits the ground?

42. (a) Show that $(2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10}$ is purely imaginary.

OR

(b) Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using integration

43. (a) Show that the value of

$$\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right) \text{ is } \frac{\pi}{3}$$

OR

(b) The parabolic communication antenna has a focus at 2 mts. distance from the vertex of the antenna. Show that the width of the antenna 3 mts. from the vertex is $4\sqrt{6}$ mts.

44. (a) Solve the differential equation $\frac{dy}{dx}+\frac{y}{x}=\sin x$.

OR

(b) Verify whether the following compound proposition is tautology or contradiction or contingency. $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$

45. (a) Prove by using vector method that cos (A – B) = cos A cos B + sin A sin B.

OR

(b) Prove that among all the rectangles of the given perimeter, the square has the maximum area.

46. (a) Find the eccentricity, foci, vertices and centre for the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and draw the rough diagram.

OR

(b) The cumulative distribution function of a discrete random variable is given by :

$$F(x) = \begin{cases} 0 & \text{for } -\infty < x < 0 \\ \frac{1}{2} & \text{for } 0 \le x < 1 \\ \frac{3}{5} & \text{for } 1 \le x < 2 \\ \frac{4}{5} & \text{for } 2 \le x < 3 \\ \frac{9}{10} & \text{for } 3 \le x < 4 \\ 1 & \text{for } 4 \le x < \infty \end{cases}$$

Find (i) The probability mass function (ii) $P(x < 3)$ and (iii) $P(x \ge 2)$

47. (a) Show that the area between the parabola $y^2 = 16x$ and its latus rectum (using integration) is $\frac{128}{3}$.

OR

(b) Show that the Cartesian equation of the plane passing through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

❁❁❁❁❁

Answers

PART - I

1. (d) $\frac{4}{5}$
2. (d) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$
3. (b) imaginary axis
4. (a) 0
5. (d) -4
6. (a) $\frac{\pi}{6}$
7. (a) $x^2 + y^2 - 6y - 7 = 0$
8. (b) $\frac{\sqrt{5}}{2}$
9. (c) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$
10. (b) 1
11. (b) 2.5
12. (c) $\frac{\pi}{2}$
13. (b) $\frac{1}{5}$
14. (c) $\frac{5}{2}$
15. (c) $\frac{8}{3}$
16. (d) 1
17. (b) $\frac{d^2y}{dx^2}+y=0$
18. (b) $\frac{1}{2}$
19. (b) 2
20. (b) Multiplication

PART - II

21.

$$\begin{aligned} \text{When } x &= 3 \text{ and } dx = 0.02, \\ df &= (2x+3)dx \\ &= [2(3)+3]\,(0.02) \\ df &= (6+3)\,(0.02) \\ &= 9(0.02) = 0.18 \end{aligned}$$

22.

$$\begin{aligned} \alpha+\beta &= -5;\ \alpha\beta = 6 \\ \alpha^2+\beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta \\ &= (-5)^2 - 2(6) = 25 - 12 = 13 \end{aligned}$$

Hence proved.

23. $\sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

24. Comparing the given lines with the general Cartesian equations of straight lines,

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3} \text{ and}$$

$$\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_3}{d_3}$$

we find $(b_1, b_2, b_3) = (2, 1, -2)$ and $(d_1, d_2, d_3) = (4, -4, 2)$. Therefore, the acute angle between the two straight lines is