PART – I

ENGINEERING MATHEMATICS

(COMMON TO ALL CANDIDATES)

UNIT-I

DETERMINANTS AND MATRICES

SOLVING SYSTEM OF EQUATIONS – RANK OF THE MATRIX – EIGENVALUES AND EIGENVECTORS – REDUCTION OF QUADRATIC FORM TO CANONICAL FORM.

DEFINITION OF A MATRIX

- Rectangular array of real numbers
- *m* rows by *n* columns
- Named using capital letters
- First subscript is row, second subscript is column

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

TERMINOLOGY

- A matrix with m rows and n columns is called a matrix of order $m \times n$.
- A square matrix is a matrix with an equal number of rows and columns. Since the number of rows and columns are the same, it is said to have order n.

- The main diagonal of a square matrix are the elements from the upper left to the lower right of the matrix.
- A row matrix is a matrix that has only one row.
- A column matrix is a matrix that has only one column.
- A matrix with only one row or one column is called a vector.

CONVERTING SYSTEMS OF LINEAR EQUATIONS TO MATRICES

Each equation in the system becomes a row. Each variable in the system becomes a column. The variables are dropped and the coefficients are placed into a matrix. If the right hand side is included, it's called an augmented matrix. If the right hand side isn't included, it's called a coefficient matrix.

The system of linear equations ...

$x + y - z = 1$

$3x - 2y + z = 3$

$4x + y - 2z = 9$

becomes the augmented matrix ...

x y z rhs

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 3 \\ 4 & 1 & -2 & 9 \end{array}\right]$$

MATRIX ADDITION AND SUBTRACTION

Two matrices A and B can be added or subtracted if and only if their dimensions are the same (i.e. both matrices have the same number of rows and columns.)

Take:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

ADDITION

If A and B above are matrices of the same type then the sum is found by adding the corresponding elements $a_{ij} + b_{ij}$.

Here is an example of adding A and B together.

$$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 2 & 0 & 5 \end{pmatrix}$$

SUBTRACTION

If A and B are matrices of the same type then the subtraction is found by subtracting the corresponding elements $a_{ij} - b_{ij}$.

Here is an example of subtracting matrices.

$$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

MATRIX MULTIPLICATION

DEFINITION:

When the number of columns of the first matrix is the same as the number of rows in the second matrix then matrix multiplication can be performed.

Here is an example of matrix multiplication for two 2 × 2 matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{pmatrix}$$

Here is an example of matrix multiplication for two 3 × 3 matrices.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{pmatrix}$$

Now look at the n × n matrix case, Where A has dimensions m × n, B has dimensions n × p. Then the product of A and B is the matrix C, which has dimensions m× p. The ij^{th} element of matrix C is found by multiplying the entries of the i^{th} row of A with the corresponding entries in the j^{th} column of B and summing the n terms. The elements of C are:

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} = \sum_{j-1}^{n} a_{1j}b_{j1}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \ldots + a_{1n}b_{n2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_{mp} = a_{m1}b_{1p} + a_{m2}b_{2p} + \ldots + a_{mn}b_{np}$$

THE DETERMINANT OF A MATRIX

Determinants play an important role in finding the inverse of a matrix and also in solving systems of linear equations. In the following we assume we have a square matrix (m = n). The determinant of a matrix A will be denoted by det(A) or |A|. Firstly the determinant of a 2 × 2 and 3 × 3 matrix will be introduced, then the n ×n case will be shown.

DETERMINANT OF A 2 × 2 MATRIX

Assuming A is an arbitrary 2×2 matrix A, where the elements are given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Solving system of equations

Systems with three equations and three variables can also be solved using matrices and row reduction. First, arrange the system in the following form:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

where $a_{1,2,3,}$ $b_{1,2,3,}$ and $c_{1,2,3}$ are the x , y , and z coefficients, respectively, and $d_{1,2,3}$ are constants.

Next, create a 3×4 matrix, placing the x coefficients in the 1st column, the y coefficients in the 2nd column, the z coefficients in the 3rd column, and the constants in the 4th column, with a line separating the 3rd column and the 4th column:

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

This is equivalent to writing

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

which is equivalent to the original three equations (check the multiplication yourself).

Finally, row reduce the 3×4 matrix using the elementary row operations. The result should be the identity matrix on the left side of the line and a column of constants on the right side of the line. These constants are the solution to the system of equations:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$$

If the system row reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 0 & e_1 \end{array}\right]$$

then the system is inconsistent. If the system row reduced to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 0 & 0 \end{array}\right]$$

then the system has multiple solutions.

Example

Solve the following system:

$5x + 3y = 2z - 4$

$2x + 2z + 2y = 0$

$3x + 2y + z = 1$

Solution

First, arrange the equations:

$5x + 3y - 2z = -4$

$2x + 2y + 2z = 0$

$3x + 2y + 1z = 1$

Next, form the 3×4 matrix:

$$\left[\begin{array}{ccc|c} 5 & 3 & -2 & -4 \\ 2 & 2 & 2 & 0 \\ 3 & 2 & 1 & 1 \end{array}\right]$$

Finally, row reduce the matrix:

$$\left[\begin{array}{ccc|c} 5 & 3 & -2 & -4 \\ 2 & 2 & 2 & 0 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{switch } R_1 \text{ and } R_2} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 5 & 3 & -2 & -4 \\ 3 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \times \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 5 & 3 & -2 & -4 \\ 3 & 2 & 1 & 1 \end{array}\right]$$