SURA'S

Mathematics

11th Standard

VOLUME - I & II

FREE Practice Workbook

Based on the Updated New Textbook

Salient Features

- Complete solutions to Textbook Exercises.
- Exhaustive Additional Questions & Answers in all chapters.
- Govt. Model Question Paper-2018 ***[Govt. MQP-2018]***, First Mid-Term Test - 2018 ***[First Mid-2018]***, Quarterly Exam - 2018 ; 2019 & 2023 ***[QY-2018; 2019 & 2023]***, Half yearly Exam - 2018 ; 2019 & 2023 ***[HY - 2018 ; 2019 & 2023]***, Public Exam March - 2019, 2020, May - 2022, April - 2023 & March - 2024, ***[Mar - 2019, 2020, May -2022, April-2023 & Mar.- 2024]***, Instant Supplementary Exam June-2019, August-2022, July-2023 & 2024 ***[June-2019, Aug.-2022, July-2023 & '24]***, Govt. Supplementary Exam September 2020 & 2021 ***[Sep-2020 & 2021]*** and Common Revision Test ***[CRT - 2022]*** Questions are incorporated in the appropriate sections.
- Quarterly Exam September - 2024 Question Paper is given with answers.

SURA PUBLICATIONS
Chennai

Code No : SG 262

Author :
Mr.H.S.Raj *M.Sc.,B.Ed.,*
Chennai.

NOTE FROM PUBLISHER

It gives me great pride and pleasure in bringing to you **Sura's Mathematics Guide Volume I and Volume II** for 11th Standard. A deep understanding of the text and exercises is rudimentary to have an insight into Mathematics. The students and teachers have to carefully understand the topics and exercises.

Sura's Mathematics 11th Standard Guide encompasses all the requirements of the students to comprehend the text and the evaluation of the textbook.

- It will be a teaching companion to teachers and a learning companion to students.
- Exhaustive additional Questions and Answers have been given to help students practice and learn effectively all the sections of the textbook.
- In order to learn effectively, I advise students to learn the subject sectionwise and practice the exercises given.

Though these salient features are available in our Sura's Mathematics Guide 11th Standard, I cannot negate the indispensable role of the teachers in assisting the student to understand the Mathematics.

I sincerely believe this guide satisfies the needs of the students and bolsters the teaching methodologies of the teachers.

I pray the almighty to bless the students for consummate success in their examinations.

Subash Raj, B.E., M.S.
- Publisher
Sura Publications

All the Best

Head Office:
Sura Publications
1620, 'J' Block, 16th Main Road,
Anna Nagar, **Chennai - 600 040.**
Phone : 044 - 4862 9977, 044 - 4862 7755.
E-mail : orders@surabooks.com
Website : www.surabooks.com

For Orders Contact
80562 94222
81242 01000
81243 01000
96001 75757
78718 02000
98409 26027

09-11-2024

CONTENT

SYLLABUS

Chapter No	Title	Month
1.	Sets, Relations and Functions	**June**
2.	Basic Algebra (2.1 - 2.5)	
2.	Basic Algebra (2.6 - 2.10)	**July**
3.	Trigonometry	
4.	Combinatorics and Mathematical Induction (4.1 - 4.4)	
I MID TERM TEST (June, July)		
4.	Combinatorics and Mathematical Induction (4.5 - 4.6)	**August**
5.	Binomial Theorem, Sequences And Series	
6.	Two Dimensional Analytical Geometry (6.1 - 6.3)	
6.	Two Dimensional Analytical Geometry (6.4 - 6.5)	**Sep.**
QUARTERLY EXAMINATION (June to September)		

Chapter No	Title	Month
7.	Matrices and Determinants	**October**
8.	Vector Algebra	
9.	Differential Calculus - Limits and Continuity	**Nov.**
10.	Differential Calculus - Differentiability and Methods of Differentiation	
	II MID TERM TEST (October, November)	
11.	Integral Calculus	**Dec.**
12.	Introduction to probability theory	
HALFYEARLY EXAMINATION (June to December)		

TO ORDER WITH US

SCHOOLS and TEACHERS:

We are grateful for your support and patronage to **'SURA PUBLICATIONS'**
Kindly prepare your order in your School letterhead and send it to us.
For Orders contact: 81242 01000 / 81243 01000

DIRECT DEPOSIT

A/c Name	:	**Sura Publications**
Our A/c No.	:	**36550290536**
Bank Name	:	**STATE BANK OF INDIA**
Bank Branch	:	Padi
IFSC	:	SBIN0005083

A/c Name	:	**Sura Publications**
Our A/c No.	:	**21000210001240**
Bank Name	:	**UCO BANK**
Bank Branch	:	Anna Nagar West
IFSC	:	UCBA0002100

A/c Name	:	**Sura Publications**
Our A/c No.	:	**6502699356**
Bank Name	:	**INDIAN BANK**
Bank Branch	:	Asiad Colony
IFSC	:	IDIB000A098

A/c Name	:	**Sura Publications**
Our A/c No.	:	**1154135000017684**
Bank Name	:	**KVB BANK**
Bank Branch	:	Anna Nagar
IFSC	:	KVBL0001154

A/c Name	:	**Sura Publications**
Our A/c No.	:	**13240200032412**
Bank Name	:	**FEDERAL BANK**
Bank Branch	:	Anna Nagar
IFSC	:	FDRL0001324

A/c Name	:	**Sura Publications**
Our A/c No.	:	**50200031530945**
Bank Name	:	**HDFC BANK**
Bank Branch	:	Cenotaph Road, Teynampet
IFSC	:	HDFC0001216

After Deposit, please send challan and order to our address.
email to : orders@surabooks.com / Whatsapp : 81242 01000.

For Google Pay : 98409 26027   **For PhonePe : 98409 26027**

DEMAND DRAFT / CHEQUE

Please send Demand Draft / cheque in favour of **'SURA PUBLICATIONS'** payable at **Chennai**. The Demand Draft / cheque should be sent with your order in School letterhead.

STUDENTS :

Order via Money Order (M/O) to

SURA PUBLICATIONS
1620, 'J' Block, 16th Main Road, Anna Nagar, Chennai - 600 040.
Phone : 044-4862 9977, 044-4862 7755.
Mobile : 96001 75757 / 81242 01000 / 81243 01000.
Email : orders@surabooks.com **Website :** www.surabooks.com

Volume - I

MATHEMATICS

enquiry@surabooks.com PH: 8124201000 / 8124301000

CONTENTS

01 Sets, Relations and Functions

MUST KNOW DEFINITIONS

A set is a collection of well defined objects.

Type of sets

Empty set	:	A set containing no element.
Finite set	:	The number of elements in the set is finite.
Infinite set	:	The number of elements in the set is not finite.
Singleton set	:	A set containing only one element.
Equivalent set	:	Two sets having same number of elements.
Equal sets	:	Two sets exactly having the same elements.
Subset	:	A set X is a subset of Y if every element of X is also an element of Y. $(X \subseteq Y)$
Proper subset	:	X is a proper subset of Y if $X \subset Y$ and $X \neq Y$.
Power set	:	The set of all subsets of A is the power set of A.
Universal set	:	The set contains all the elements under consideration

Algebra of sets

Union	:	The union of two sets A and B is the set of elements which are either in A or in B $(A \cup B)$
Intersection	:	The intersection of two sets A and B is the set of all elements common to both A and B $(A \cap B)$.
Complement of a set	:	The complement of a set is the set of all elements of U (Universal set) that are not elements of A. (A') Set different$(A\backslash B)$ or $(A - B)$
Difference of two sets	:	The difference of the two sets A and B is the set of all elements belonging to A but not to B. Set different$(A\backslash B)$ or $(A - B)$
Disjoint sets	:	Two sets A and B are said to be disjoint if there is no element common to both A and B.
Open interval	:	The set $\{x: a < x < b\}$ is called an open interval and denoted by (a, b)
Closed interval	:	The set $\{x: a \leq x \leq b\}$ is called a closed interval and denoted by $[a, b]$
Neighbourhood of a point	:	Let a be any real number. Let $\in > 0$ be arbitrarily small real number. Then $(a - \in, a + \in)$ is called an "$\in$" neighbourhood of the point a and denoted by $N_{a,\in}$

Cartesian product of sets : The set of all ordered pairs (a, b) such that $a \in$ A and $b \in$ B is called the cartesian product of A and B and is denoted by A × B.

Types of relation

Reflexive : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Symmetric : A relation R on a set A is said to be symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in$ A.

Transitive : A relation R on a set A is said to be transitive if $(a, b) \in$ R and $(b, c) \in$ R $\Rightarrow (a, c) \in$ R for all $a, b, c \in$ A.

Equivalent : A relation R on a set A is said to be equivalence relation if it is reflexive, symmetric and transitive.

Function : A function f from a set A to a set B is a rule which assigns to each element of A, a unique element of B.
If f: A → B, then A is the domain, B is the co-domain.

Types of algebraic functions

Identity function : A function that associates each real number to itself.

Absolute value function : The function $f(x)$ defined by $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Constant function : A function $f(x)$ defined by $f(x) = k$ where k is a real number.

Greatest integer function : The greatest integer function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ for all $x \in$ R.

Smallest integer function : The smallest integer function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lceil x \rceil$ for all $x \in$ R.

Signum function : The function f defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Polynomial function : The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$ where $a_0, a_1, \ldots, a_n$ are constants.

Rational function : The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ and $p(x), q(x)$ are polynomial.

Algebra of functions

Addition : If $f: D_1 \to$ R and $g: D_2 \to$ R, then their sum $f + g: D_1 \cap D_2 \to$ R such that $(f + g)(x) = f(x)\ \ g(x)$ for all $x \in D_1 \cap D_2$.

Subtraction : If $f_1: D_1 \to$ R and $g: D_2 \to$ R, then their difference $f - g: D_1 \cap D_2 \to$ R such that $(f - g)(x) = f(x) - g(x)$ for all $x \in D_1 \cap D_2$.

Product : If $f_1: D_1 \to$ R and $g: D_2 \to$ R, then their product $f \cdot g: D_1 \cap D_2 \to$ R such that $(fg)(x) = f(x) \cdot g(x)$ for all $x \in D_1 \cap D_2$.

Quotient : If $f_1: D_1 \to$ R and $g: D_2 \to$ R, then their quotient $\frac{f}{g}: D_1 \cap D_2 - \{x : g(x) = 0\} \to$ R such that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ such that for all $x \in D_1 \cap D_2 - \{x : g(x) = 0\}$.

Composition of functions : If f: A → B and g: B → C then gof : A → C defined by $gof(x) = g[f(x)]$ for all $x \in$ A.

Kinds of functions

One-one : A function $f: A \rightarrow B$ is said to be a one-one function (injection) if different elements of A have different images in B.

Onto : A function $f: A \rightarrow B$ is said to be an onto (surjection) function if every element of B is the image of some element of A.

Bijection : A function $f: A \rightarrow B$ is a bijection if one-one as well as onto.

Inverse of a function : Let $f: A \rightarrow B$ be a bijection. Then $g: B \rightarrow A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $g(y) = x$ is called the inverse of f, and it denoted as f'.

Formulae to remember

Demorgan's laws :
1. $(A \cup B)' = A' \cap B'$
2. $(A \cap B) = A' \cup B'$
3. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$
4. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

Reflexive : aRa for all $a \in A$

Symmetric : $aRb \Rightarrow bRa$ for all $a, b \in A$

Transitive : $aRb, bRc \Rightarrow aRc$ for all $a, b, c \in A$

Antisymmetric : aRb and $bRa \Rightarrow a = b$ for all $a, b \in A$ $\quad A \Delta B = (A \backslash B) \cup (B \backslash A)$

One-one function : If $f: A \rightarrow A$ then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in A$

Onto function : Co-domain = Range.

If a set has n elements, then total number of subsets is 2^n.

TEXTUAL QUESTIONS

Exercise 1.1

1. Write the following in roster form.

(i) $\{x \in \mathbb{N} : x^2 < 121$ **and** x **is a prime**$\}$.

(ii) the set of all positive roots of the equation $(x - 1)(x + 1)(x^2 - 1) = 0$.

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

(iv) $\left\{x : \dfrac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

Sol : **(i)** $\{x \in \mathbb{N} : x^2 < 121$ **and** x **is a prime**$\}$.

Let $A = \{x \in \mathbb{N}: x^2 < 121$, and x is a prime$\}$

$A = \{2, 3, 5, 7\}$.

(ii) the set of all positive roots of the equation $(x - 1)(x + 1)(x^2 - 1) = 0$.

Let B = {the set of positive roots of the equation $(x - 1)(x + 1)(x^2 - 1) = 0$}

$$(x - 1)(x + 1)(x - 1)(x + 1) = 0$$
$$(x + 1)^2 (x - 1)^2 = 0$$
$$(x + 1)^2 = 0 \text{ or } (x - 1)^2 = 0$$
$$x + 1 = 0 \text{ or } x - 1 = 0$$
$$x = -1 \text{ or } x = 1$$
$$\Rightarrow \quad x = 1, -1$$
$$B = \{1\}.$$

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

Let $C = \{x \in \mathbb{N}: 4x + 9 < 52\}$

$\Rightarrow C = \{x \in \mathbb{N}: 4x < 52 - 9\}$

$\Rightarrow C = \{x \in \mathbb{N}: 4x < 43\}$

$\Rightarrow C = \left\{x \in \mathbb{N}: x < \dfrac{43}{4}\right\}$

$\Rightarrow C = \{x \in \mathbb{N}: x < 10.75\}$

$\Rightarrow C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(iv) $\left\{x : \dfrac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$.

Let $D = \left\{x : \dfrac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

$\Rightarrow D = \{x: x - 4 = 3x + 6, x \in \mathbb{R}\}$

$\Rightarrow D = \{x: -4 - 6 = 3x - x, x \in \mathbb{R}\}$

$\Rightarrow D = \{x: 2x = -10, x \in \mathbb{R}\}$

$\Rightarrow D = \{x: x = -5, x \in \mathbb{R}\}$

$\Rightarrow D = \{-5\}$

2. Write the set $\{-1, 1\}$ **in set builder form.**

Sol : Let $P = \{-1, 1\}$

$\Rightarrow P = \{x : x$ is a root of $x^2 - 1 = 0\}$

$\Rightarrow P = \{x : x^2 - 1 = 0, x \in \mathbb{R}\}$

3. State whether the following sets are finite or infinite.

(i) $\{x \in \mathbb{N} : x$ **is an even prime number}**

(ii) $\{x \in \mathbb{N} : x$ **is an odd prime number}**

(iii) $\{x \in \mathbb{Z} : x$ **is even and less than 10}**

(iv) $\{x \in \mathbb{R} : x$ **is a rational number}**

(v) $\{x \in \mathbb{N} : x$ **is a rational number}**

Sol : **(i)** $\{x \in \mathbb{N} : x$ **is an even prime number}**

Let A = $\{x \in \mathbb{N}$: x is an even prime number}

$\Rightarrow$ A = {2} $\Rightarrow$ A is a finite set.

(ii) $\{x \in \mathbb{N} : x$ **is an odd prime number}**

Let B = $\{x \in \mathbb{N}$: x is an odd prime number}

$\Rightarrow$ B = {1, 3, 5, 7, 11, ...}

$\Rightarrow$ B is an infinite set.

(iii) $\{x \in \mathbb{Z} : x$ **is even and less than 10}**

Let C = $\{x \in \mathbb{Z}$: x is even and < 10}

$\Rightarrow$ C = {...–4, –2, 0, 2, 4, 6, 8}. C is a infinite set.

(iv) $\{x \in \mathbb{R} : x$ **is a rational number}**

Let D = $\{x \in \mathbb{R}$: x is a rational number}

$\Rightarrow$ D = {set of all rational number}

$\Rightarrow$ D is an infinite set.

(v) $\{x \in \mathbb{N} : x$ **is a rational number}**

Let $\mathbb{N}$ = $\{x \in \mathbb{N}$: x is a rational number}

$$\Rightarrow \mathbb{N} = \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \ldots \infty\right\}$$

$\Rightarrow \mathbb{N}$ is an infinite set.

4. By taking suitable sets A, B, C, verify the following results:

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

(iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$

(v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$

Sol : **(i)** $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Let A = {1, 2, 3}, B = (4, 5, 6, 7}

C = {3,4,5,9}

and U = {1, 2, 3, 4, 5, 6, 7, 9}

LHS = $A \times (B \cap C)$

= $A \times$ {4, 5} [$\because B \cap C$ = {4,5}]

= {1, 2, 3} × {4, 5}

= {(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)} ...(1)

A × B = {1, 2, 3} × {4, 5, 6, 7}

= {(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)}

A × C = {1, 2, 3} × {3, 4, 5, 9}

= {(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)}

RHS = $(A \times B) \cap (A \times C)$

= {(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)} ...(2)

From (1) and (2), LHS = RHS. Hence verified.

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$(B \cup C)$ = {3, 4, 5, 6, 7, 9}

Now, $A \times (B \cup C)$ = {1, 2, 3} × {3, 4, 5, 6, 7, 9}

= {(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3, 9)} ...(1)

Now A × B = {1, 2, 3} × {4, 5, 6, 7}

= {(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)}

A × C = {1, 2, 3} × {3, 4, 5, 9}

= {(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)}

RHS $(A \times B) \cup (A \times C)$

= {(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3,9)} ...(2)

From (1) & (2), LHS = RHS Hence verified

(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

(A × B) = {(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)}

(B × A) = {(4, 1) (4, 2) (4, 3) (5, 1) (5, 2) (5, 3) (6, 1) (6, 2) (6, 3) (7, 1) (7, 2) (7, 3)}

LHS = $(A \times B) \cap (B \times A)$ = { } ...(1)

$(A \cap B)$ = { }, $(B \cap A)$= { }

$\therefore$ RHS = $(A \cap B) \times (B \cap A)$ ={ }...(2)

From (1) and (2), LHS = RHS

(iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$

B – A = {4, 5, 6, 7}

LHS = C – (B – A) = {3, 9} ...(1)

$C \cap A$ = {3}

B′ = {1, 2, 3, 9}

$C \cap B'$ = {3, 9}

RHS = $(C \cap A) \cup (C \cap B')$

= {3, 9} ...(2)

From (1) and (2), LHS = RHS

(v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

B – A = {4, 5, 6, 7}

$(B - A) \cap C$ = {4, 5} ...(1)

$B \cap C$ = {4, 5}

$(B \cap C) - A$ = {4, 5} ...(2)

$C - A = \{4, 5, 9\}$

$B \cap (C - A) = 4, 5\}$ ...(3)

From (1), (2) and (3),

$(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$.

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$

$B - A = \{4, 5, 6, 7\}$

$(B - A) \cup C = \{3, 4, 5, 6, 7, 9\}$ ...(1)

$B \cup C = \{3, 4, 5, 6, 7, 9\}$

$A - C = \{1, 2\}$

$(B \cup C) - (A - C) = \{3, 4, 5, 6, 7, 9\}$ ...(2)

From (1) and (2), $(B - A) \cup C = (B \cup C) - (A - C)$

Hence verified.

5. Justify the trueness of the statement "*An element of a set can never be a subset of itself*".

Sol : Let $P = \{a, b, c, d\}$.

Each and every element of the set P can be a subset of the set itself

Eg : $\{a\}, \{b\}, \{c\}, \{d\}$.

Hence, the given statement is not true.

6. If $n(p(A)) = 1024$, $n(A \cup B) = 15$ and $n(P(B)) = 32$, then find $n(A \cap B)$.

Sol : Given $n(P(A)) = 1024 = 2^{10}$ [$\therefore$ If $n(A) = n$, then $n(P(A)) = 2^n$]

$\Rightarrow n(A) = 10$

$n(P(B)) = 32 = 2^5$

$\Rightarrow n(B) = 5$.

We know that,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow 15 = 10 + 5 - n(A \cap B)$

$\Rightarrow n(A \cap B) = 0$.

7. If $n(A \cap B) = 3$ and $n(A \cup B) = 10$, then find $n(P(A \Delta B))$ [Qy. - 2018; CRT - 2022; Qy. - 2023]

Sol : We know that $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$ if A and B are not disjoint.

$\Rightarrow n(A - B) + n(B - A) = n(A \cup B) - n(A \cap B)$

$\Rightarrow n(A \Delta B) = 10 - 3$

$\Rightarrow \therefore n(A \Delta B) = 7$

$\therefore n[P(A \Delta B)] = 2^7 = 128$.

8. For a set A, A × A contains 16 elements and two of its elements are (1, 3) and (0, 2). Find the elements of A.

Sol : Since A × A contains 16 elements, then A must have 4 elements

$\Rightarrow n(A) = 4$.

The elements of A × A are (1, 3) and (0, 2)

$\therefore$ The possibilities of elements of A are $\{0, 1, 2, 3\}$

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1)$ $(y, 2)$ $(z, 1)$ are in A × B, find A and B, where x, y, z are distinct elements. [Hy. - 2018]

Sol : Given $A \times B = \{(x, 1) (y, 2) (z, 1)\}$

Since $n(A) = 3$ and $n(B) = 2$,

A × B will have 6 elements.

The remaining elements of A × B will be $(x, 2)$ $(y, 1)$ $(z, 2)$

$\therefore A \times B = \{(x, 1) (y, 2) (z, 1) (x, 2)(y, 1) (z, 2)\}$

$\therefore A = \{x, y, z\}$ and $B = \{1, 2\}$

10. If A × A has 16 elements, $S = \{(a, b) \in A \times A: a < b\}$; (–1, 2) and (0, 1) are two elements of S, then find the remaining elements of S. [Qy. - 2018; June - 2019; July - 2023 & '24]

Sol : $n(A \times A) = 16 \Rightarrow n(A) = 4$.

Given $S = \{(a, b) \in A \times A: a < b\}$

$\therefore A = \{-1, 0, 1, 2\}$.

$A \times A = \{(-1, -1) (-1, 0) (-1, 1) (-1, 2)(0, -1),(0,0) (0,1) (0,2) (1, -1) (1, 0)(1, 1) (1, 2) (2, -1)(2, 0) (2, 1) (2, 2)\}$

Now, $S = \{(-1, 0) (-1, 1) (-1, 2) (0, 1) (0, 2) (1,2)\}$

$\therefore$ The remaining elements of S are (–1, 0) (–1, 1) (0, 2) (1,2)

Exercise 1.2

1. Discuss the following relations for reflexivity, Symmetric and Transitive :

(i) The relation R defined on the set of all positive integers by "mRn if m divides n".

(ii) Let P denote the set of all straight lines in a plane. The relation R defined by "lRm if l is perpendicular to m". [Qy. - 2019]

(iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".

(v) On the set of natural numbers the relation R defined by "xRy if $x + 2y = 1$".

Sol : (i) The relation R defined on the set of all positive integers by "mRn" if m divides n".

Given relation is "mRn if m divides n".

Reflexive : mRm since m divides m for all positive integers m.

$\therefore$ R is reflexive.

Symmetric : mR$n \neq n$Rm.
m divides $n \Rightarrow$ 4 divides 2 $\neq$ 2 divides 4.
$\therefore$ R is not symmetric

Transitive : mRn and nR$p \Rightarrow m$Rp.
m divides n and n divides p then m divides p.
[For eg : 4 divides 2, 2 divides 8, then4 divides 8]
$\therefore$ R is transitive.

$\therefore$ R is reflexive, not symmetric and transitive.

(ii) Let *P* denote the set of all straight lines in a plane. The relation *R* defined by "*l*R*m* if *l* is perpendicular to *m*".

Let $l, m, n \in p$.

Reflexive : We cannot say l is perpendicular to l itself.
$\therefore l \not{R}\ l \Rightarrow$ R is not reflexive.

Symmetric : lR$m \Rightarrow m$Rl
l is perpendicular to m = m is perpendicular to l
$\therefore$ R is symmetric

Transitive : lRm and mR$n \neq l$Rn.
l is perpendicular to m and m is perpendicular to n.
$\Rightarrow$ l is not perpendicular to n.
$\therefore$ R is not transitive.
$\Rightarrow$ R is only symmetric.

(iii) Let A be the set consisting of all the members of a family. The relation R defined by "*a*R*b* if *a* is not a sister of *b*".

Given relation is "aRb if a is not a sister of b". and $a, b, c \in$ A.

Reflexive : aR$a \Rightarrow a$ is not a sister of a
$\therefore$ R is reflexive.

Symmetric : aR$b \neq b$Ra
a is not a sister of b but b may be a sister of a
$\therefore$ R is not symmetric.

Transitive : aRb and bR$c \neq a$Rc
a is not a sister of b and b is not a sister of c, but a may be a sister of c.
$\therefore$ R is not transitive.
$\therefore$ R is only reflexive.

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "*a*R*b* if *a* is not a sister of *b*".

Given relation is aRb if a is not a sister of b.
Let $a, b, c \in$ A.

Reflexive : aR$a \Rightarrow a$ is not a sister of a
$\therefore$ R is reflexive.

Symmetric : aR$b \Rightarrow b$Ra
a is not a sister of $b \Rightarrow b$ is not a sister of a.
$\therefore$ R is symmetric.

Transitive : aRb and bR$c \neq a$Rc
a is not a sister of b, b is not a sister of c does not imply a is not a sister of c. [Eg :Mother is not a sister of daughter, daughter is not a sister of chithi, but mother is a sister of chithi.]
$\therefore$ R is not transitive.
$\therefore$ R is reflexive, symmetric and not transitive.

(v) On the set of natural numbers, the relation R is defined by "*x*R*y* if *x* + 2*y* = 1".

The relation R is defined by xRy if $x + 2y = 1$ for $x, y \in$ N.

Reflexive : Let $x \in \mathbb{N}$
xR$x \Rightarrow x + 2x = 1$
$\Rightarrow 3x = 1 \quad \Rightarrow x = \frac{1}{3} \notin \mathbb{N}$
$\therefore$ R is not reflexive.

Symmetric : R is an empty relation.
$\therefore$ R is symmetric (by definition)

Transitive : R is an empty relation.
$\therefore$ R is transitive. (by definition)

$\therefore$ R is an empty set; not reflexive; symmetric; transitive.

2. Let X = {*a*, *b*, *c*, *d*}, and R = {(*a*, *a*) (*b*, *b*) (*a*, *c*)}. Write down the minimum number of ordered pairs to be included to R to make it **[Qy. - 2023]**

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : Given X = {a, b, c, d} and R = {(a, a) (b, b) (a, c)}

(i) To make the relation R reflexive we must have (c, c) and (d, d) $\in$ R
$\therefore$ Minimum number of ordered pairs to be included to R to make it reflexive is (c, c) and (d, d)

(ii) To make R symmetric, we must have (c, a) $\in$ R
$\therefore$ Minimum number of ordered pairs to be included to R to make it symmetric is (c, a).

(iii) R is transitive.
$\therefore$ Nothing need to be included.

(iv) Minimum number of ordered pairs to be included to make R equivalence is (c, c) (d, d) (c, a).

3. Let A = {a, b, c}, and R = {(a, a) (b, b) (a, c)}. Write down the minimum number of ordered pairs to be included to R to make it [July - 2023]

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : (i) The ordered pairs (c, c) should be included to R to make it reflexive.

∴ Minimum number of ordered pair is (c, c)

(ii) The ordered pairs (c,a) should be included to R to make it symmetric.

∴ Minimum number of ordered pair is (c, a).

(iii) The relation is transitive.

∴ Nothing needs to included.

(iv) The ordered pairs (c, c) and (c, a) should be included to R to make it equivalence.

∴ Minimum number of ordered pairs are (c, c) and (c, a).

4. Let P be the set of all triangles in a plane and R be the relation defined on P as aRb if a is similar to b. Prove that R is an equivalence relation.

Sol : Let P be the set of all triangles in a plane and R is defined as aRb if a is similar to b.

Let $a, b, c \in$ P.

Reflexive : aR$a \Rightarrow a$ is similar to a for all $a \in$ P.
∴ R is reflexive

Symmetric : aR$b \Rightarrow b$Ra
a is similar to b
$\Rightarrow b$ is similar to a for all $a, b \in$ P.

Transitive : aRb, and bR$c \Rightarrow a$Rc.
a is similar to b and b is similar to c
$\Rightarrow a$ is similar to c.

Hence R is an equivalence relation.

5. On the set of natural numbers let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check whether it is

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : Given relation is $2a + 3b = 30$ for all $a, b \in \mathbb{N}$.

$2a + 3b = 30 \Rightarrow 2a = 30 - 3b$

$\Rightarrow a = \dfrac{30-3b}{2}$

a	12	9	6	3
b	2	4	6	8

∴ The list of ordered pairs are (12, 2) (9, 4) (6, 6) (3, 8)

(i) **Reflexive** : (12, 12) $\notin$ R $\Rightarrow$ R is not reflexive.

(ii) **Symmetric** : (9, 4) $\in$ R but (4, 9) $\notin \mathbb{R}$
∴ R is not symmetric

(iii) **Transitive** : Clearly R is transitive.
[∵ (a, b) (b, c) $\notin \mathbb{R}$]

(iv) R is not an equivalence relation.

6. Prove that the relation "friendship" is not an equivalence relation on the set of all people in Chennai.

Sol : Let a, b, c are people in Chennai

Reflexive : "a" is a friend of "a" $\Rightarrow a \not{R} a$.
$\Rightarrow$ R is not reflexive.

Symmetric : a is friend of $b \Rightarrow b$ is the friend of a.
∴ aR$b \Rightarrow b$R$a \Rightarrow$ R is symmetric

Transitive : a is the friend of b and b is the friend of $c \Rightarrow a$ need not be the friend of c.
∴ aR$b \Rightarrow b$R$c \neq a$R$c \Rightarrow$ R is not transitive

Hence, the relation "friendship" is not equivalent.

7. On the set of natural number let R be the relation defined by aRb if $a + b \leq 6$. Write down the relation by listing all the pairs. Check whether it is

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : The relation is defined by aRb if $a + b \leq 6$ for all a, $b \in \mathbb{N}$. $a + b \leq 6 \Rightarrow a \leq 6 - b$

a	1	1	1	1	1	2	2	2	2	3	3	3	4	4	5
b	1	2	3	4	5	1	2	3	4	1	2	3	1	2	1

∴ The list of ordered pairs are (1, 1) (1, 2) (1, 3) (1, 4) (1, 5), (2, 1), (2, 2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2),(5, 1)

(i) **Reflexive** : R is not reflexive since (4, 4) $\notin \mathbb{R}$.

(ii) **Symmetric** : (1, 5) $\in \mathbb{R} \Rightarrow$ (5, 1) $\in \mathbb{R}$
(2, 4) $\in \mathbb{R} \Rightarrow$ (4, 2) $\in \mathbb{R}$
(1, 2) $\in \mathbb{R} \Rightarrow$ (2, 1) $\in \mathbb{R}$
(1, 3) $\in \mathbb{R} \Rightarrow$ (3, 1) $\in \mathbb{R}$
(1, 4) $\in \mathbb{R} \Rightarrow$ (4, 1) $\in \mathbb{R}$
(2, 3) $\in \mathbb{R} \Rightarrow$ (3, 2) $\in \mathbb{R}$
∴ R is symmetric

(iii) **Transitive** : (3, 1) $\in \mathbb{R}$ and (1, 5) $\in \mathbb{R}$
$\Rightarrow$ (3, 5) $\notin$ R
∴ R is not transitive.

(iv) R is not an equivalence relation.

8. Let A = {a, b, c}. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?

Sol : Given A = {a, b, c}

(i) Let R = {(a, a) (b, b) (c, c)}

R is reflexive

R is symmetric [$\because (a, b) \notin \mathbb{R} \Rightarrow (b, a) \notin \mathbb{R}$] and

R is transitive [$\because (a, b)\ (b, c) \notin \mathbb{R} \Rightarrow (a, c) \notin \mathbb{R}$]

R is an equivalence relation.

This is the equivalence relation of smallest cardinality on A.

$\therefore n(\text{R}) = 3$

(ii) Let R = {(a, a) (a, b) (a, c) (b, a) (b, b) (b, c) (c, a) (c, b) (c, c)}

R is reflexive since (a, a) (b, b) and $(c, c) \in \mathbb{R}$

R is symmetric since $(a, b) \in \text{R} \Rightarrow (b, a) \in \mathbb{R}$

$(b, c) \in \mathbb{R} \Rightarrow (c, b) \in \mathbb{R}$

$(c, a) \in \mathbb{R} \Rightarrow (a, c) \in \mathbb{R}$

R is also transitive since $(a, b)\ (b, c) \in \mathbb{R}$

$\Rightarrow (a, c) \in \mathbb{R}$

Hence R is are equivalence relation of largest cardinality on A.

$\therefore n(\text{R}) = 9$

9. In the set Z of integers, define mRn if $m - n$ is divisible by 7. Prove that R is an equivalence relation. **[Sep. - 2020; QY-'24]**

Sol : As $m - m = 0$,

$m - m$ is divisible by 7 $\Rightarrow m\text{R}m$

$\therefore$ R is reflexive.

Let mRn.

Then $m - n = 7k$ for some integer k

Thus $n - m = 7(-k)$ and hence nRm

$\therefore$ R is symmetric.

Let mRn and nRp

$\Rightarrow m - n = 7k$ and $n - p = 7l$

$\Rightarrow m = 7k + n$ and

$-p = 7l - n$ for some integers k and l

so $m - p = 7k + \cancel{n} + 7l - \cancel{n}$

$\Rightarrow m - p = 7(k + l) \Rightarrow m\text{R}p$

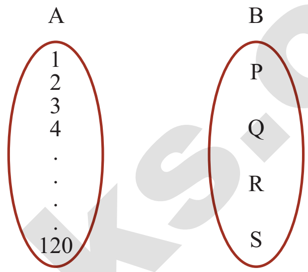
$\therefore$ R is transitive.

Thus, R is an equivalence relation.

EXERCISE 1.3

1. Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as "x related to y if the student x belongs to the section y". Is this relation a function? What can you say about the inverse relation? Explain your answer.

Sol : Given $n(\text{A}) = 120$, $n(\text{B}) = 4$



xRy is the student x belongs to the section y.

This relation is a function since every student of set A will be mapped on to some section in B.

$\therefore f$ is a function from A $\rightarrow$ B.

The inverse relation is f^{-1}: B $\rightarrow$ A.

The inverse relation is not a function since one section will have more than one student.

2. Write the values of f at –4, 1, –2, 7, 0 if

$$f(x) = \begin{cases} -x+4 & \text{if } -\infty < x \le -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2 - x & \text{if } -2 \le x < 1 \\ x - x^2 & \text{if } 1 \le x < 7 \\ 0, & \text{otherwise} \end{cases}$$

[Aug - 2022; April ; Qy. & Hy. -2023]

Sol : Now $f(-4) = 4 + 4 = 8$ [$\because f(x) = -x + 4$ when $x = -4$]

$f(1) = 1 - 1^2$ [$\because f(x) = x - x^2$ when $x = 1$]

$f(1) = 0$

$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

[$\because f(x) = x^2 - x$ when $x = -2$]

$f(7) = 0$ [$\because f(x) = 0$ when $x = 7$]

$f(0) = 0^2 - 0 = 0.$ [$\because f(x) = x^2 - x$ when $x = 0$]

$\therefore f(-4) = 8, f(1) = 0,$

$f(-2) = 6, f(7) = 0$ and $f(0) = 0$

3. Write the values of f at –3, 5, 2, –1, 0 if **[Hy. - 2019]**

$$f(x) = \begin{cases} x^2 + x - 5 & \text{if } x \in (-\infty, 0) \\ x^2 + 3x - 2 & \text{if } x \in (3, \infty) \\ x^2 & \text{if } x \in (0, 2) \\ x^2 - 3, & \text{Otherwise} \end{cases}$$

[First Mid - 2018; Sep. - 2021; CRT - 2022; QY-'24 Mar.&July - 2024]

Sol :

$f(-3) = (-3)^2 - 3 - 5 = 9 - 3 - 5 = 9 - 8 = 1$
$[\because f(x) = x^2 + x - 5 \text{ when } x = -3]$

$f(5) = 5^2 + 3(5) - 2 = 25 + 15 - 2 = 38$
$[\because f(x) = x^2 + 3x - 2 \text{ when } x = 5]$

$f(2) = 2^2 - 3 = 4 - 3 = 1$
$[\because f(x) = x^2 - 3 \text{ when } x = 2]$

$f(-1) = (-1)^2 + (-1) - 5 = \cancel{1} - \cancel{1} - 5 = -5$
$[\because f(x) = x^2 + x - 5 \text{ when } x = -1]$

$f(0) = 0^2 - 3 = -3$
$[\because f(x) = x^2 - 3 \text{ when } x = 0]$

$\therefore f(-3) = 1, f(5) = 38,$
$f(2) = 1, f(-1) = -5, f(0) = -3$

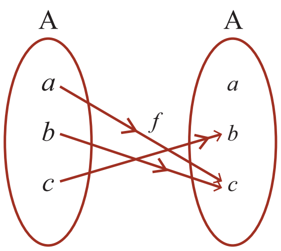
4. State whether the following relations are functions or not. If it is a function check for one-to-oneness and ontoness . If it is not a function state why?

(i) If A = $\{a, b, c\}$ and $f = \{(a, c)\ (b, c)\ (c, b)\}$: $(f : A \to A)$.

(ii) If X = $\{x, y, z\}$ and $f = \{(x, y)\ (x, z)\ (z, x)\}$: $(f : X \to X)$

Sol : (i) If A = $\{a, b, c\}$ and $f = \{(a, c)\ (b, c)\ (c, b)\}$ $(f : A \to A)$.

Given $f : A \to A$



This is a function. Since different elements of A does not have different images in A.

$\therefore f$ is not one-one.

Here

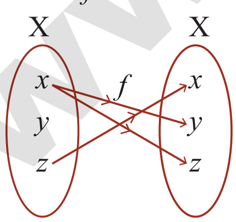
Co-domain $= \{a, b, c\}$

But Range $= \{b, c\}$

f is not onto since co-domain $\neq$ Range.

(ii) If X = $\{x, y, z\}$ and $f = \{(x, y)\ (x, z)\ (z, x)\}$: $(f : X \to X)$

Given $f : X \to X$



f is not a function since the element x have two images namely y and z.

5. Let A = {1, 2, 3, 4} and B = $\{a, b, c, d\}$. Give a function from A $\to$ B for each of the following :

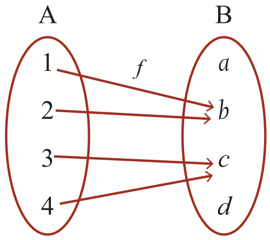
(i) neither one- to -one nor onto.

(ii) not one-to-one but onto.

(iii) one-to-one but not onto.

(iv) one-to-one and onto.

Sol : (i) neither one- to -one nor onto.



Let $f = \{(1, b)\ (2, b)\ (3, c)\ (4, c)\}$

Different elements in A does not have different images in B

$\therefore f$ is not one-one

Now, Co-domain $= \{a, b, c, d\}$,

Range $= \{b, c\}$

Co-domain $\neq$ range

$\therefore f$ is not onto.

Hence f is neither one-one and nor onto.

(ii) not one-to-one but onto.

Given A = {1, 2, 3, 4}, and B = $\{a, b, c, d\}$

Let f : A $\to$ B.

The function does not exist for not one-one but onto.

Since f : A $\to$ B, f is onto

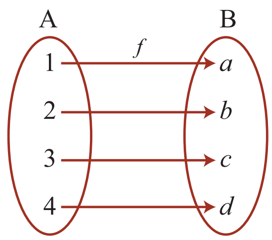
$\Rightarrow f$ must be one one since $n(A) = n(B)$

(iii) one-to-one but not onto.

The function does not exist for one-to-one but not onto.

Since f : A$\to$ B, f is one-one $\Rightarrow f$ must be onto
$[\because n(A) = n(B)]$

(iv) one-to-one and onto.



Let f: A $\to$ B defined by

$f = \{(1, a)\ (2, b)\ (3, c)\ (4, d)\}$

Here different elements have different images

$\therefore f$ is one-to-one.

Also Co-domain $= \{a, b, c, d\}$ = Range.

$\therefore f$ is onto.

$\therefore f$ is one-to-one and onto.

6. Find the domain of $\dfrac{1}{1 - 2\sin x}$.

[Sep. 2020; Hy. - 2019; Mar. - 2024]

Sol : Let $f(x) = \dfrac{1}{1 - 2\sin x}$.

When the denominator is 0,

$$1 - 2\sin x = 0 \Rightarrow 1 = 2\sin x$$

$$\Rightarrow \sin x = \frac{1}{2} \Rightarrow \sin x = \sin\frac{\pi}{6}$$

$$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbb{Z}$$

$[\because \sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n \alpha, n \in \mathbb{Z}]$

Domain of $f(x)$ is $R - \left\{n\pi + (-1)^n \frac{\pi}{6}\right\}, n \in \mathbb{Z}$

7. Find the largest possible domain of the real valued function $f(x) = \frac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$.

Sol : Given $f(x) = \frac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$.

When $x = 2, f(x) = 0$

When $x = -2, f(x) = 0$

For all the other values, we get negative value in the square root which is not possible.

$\therefore$ Domain $= \phi$

8. Find the range of the function $\frac{1}{2\cos x - 1}$.

[Govt. MQP - 2018; June - 2019; April - 2023]

Sol : Range of cosine function is $-1 \le \cos x \le 1$.

$\Rightarrow -2 \le 2\cos x \le 2$ (Multiplied by 2)

$\Rightarrow -2 - 1 \le 2\cos x - 1 \le 2 - 1$

$\Rightarrow -3 \le 2\cos x - 1 \le 1$

$$\Rightarrow \frac{-1}{3} \ge \frac{1}{2\cos x - 1} \ge \frac{1}{1} \Rightarrow \frac{-1}{3} \ge f(x) \ge 1$$

$\therefore$ Range of $f(x)$ is $\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$

9. Show that the relation $xy = -2$ is a function for a suitable domain. Find the domain and the range of the function.

Sol : Given relation is $xy = -2$.

$$\Rightarrow x = -\frac{2}{y}$$

$$\text{Now } f(x_1) = f(x_2) \Rightarrow -\frac{2}{y_1} = -\frac{2}{y_2}$$

$$\Rightarrow \frac{1}{y_1} = \frac{1}{y_2} \Rightarrow y_1 = y_2$$

$\therefore f$ is a one-one function

The element $0 \in$ the domain will not have the image.

$\therefore$ Domain $= R - \{0\}$ and Range $= R - \{0\}$.

10. If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = |x| + x$ and $g(x) = |x| - x$, find $g \circ f$ and $f \circ g$. **[Mar. - 2020]**

Sol : Given $f(x) = |x| + x$

$$= \begin{cases} x + x = 2x & \text{if } x \ge 0 \\ -x + x = 0 & \text{if } x < 0 \end{cases}$$

$$g(x) = |x| - x$$

$$= \begin{cases} x - x = 0 & \text{if } x \ge 0 \\ -x - x = -2x & \text{if } x < 0 \end{cases}$$

$$\text{Now, } fog(x) = f(g(x))$$

$$= \begin{cases} f(0) & \text{if } x \ge 0 \\ f(-2x) & \text{if } x < 0 \end{cases}$$

$$\Rightarrow fog(x) = \begin{cases} 2 \times 0 = 0 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$\therefore fog(x) = 0$ for all $x \in \mathbb{R}$.

$$\text{and } gof(x) = g(f(x)) = \begin{cases} g(2x) & \text{if } x \ge 0 \\ g(0) & \text{if } x < 0 \end{cases}$$

$$\Rightarrow gof(x) = \begin{cases} 0 \text{ if } x \ge 0 \\ 0 \text{ if } x < 0 \end{cases}$$

$\Rightarrow gof(x) = 0$ for all $x \in \mathbb{R}$.

11. If f, g, h are real valued functions defined on $\mathbb{R}$, then prove that $(f+g)oh = foh + goh$. What can you say about $fo(g+h)$? Justify your answer.

Sol : (i) Since f, g, h are functions from $\mathbb{R} \to \mathbb{R}$,

$(f+g)oh: \mathbb{R} \to \mathbb{R}$ and $foh + goh: \mathbb{R} \to \mathbb{R}$.

For any $x \in \mathbb{R}$,

$$[(f+g)oh](x) = (f+g)(h(x))$$
$$= f(h(x)) + g(h(x))$$
$$= foh(x) + goh(x)$$
$$\therefore (f+g)oh = foh + goh$$

(ii) Also $fo(g+h) = f[(g+h)(x)]$ for any $x \in \mathbb{R}$

$$= f[g(x) + h(x)] = f(g(x)) + f(h(x))$$
$$= fog(x) + foh(x).$$
$$\therefore fo(g+h) = fog(x) + foh(x).$$

12. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 5$, prove that f is a bijection and find its inverse.

[Govt.MQP & Qy. - 2018]

Sol : Let $y = 3x - 5$.

$$\Rightarrow y + 5 = 3x \Rightarrow \frac{y+5}{3} = x.$$

$$\text{Let } g(y) = \frac{y+5}{3}.$$

$$gof(x) = g(f(x)) = g(3x - 5)$$

$$= \frac{3x - \cancel{5} + \cancel{5}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

Also $fog(y) = f(g(y)) = f\left(\frac{y+5}{3}\right)$

$$= 3\left(\frac{y+5}{3}\right) - 5 = y + 5 - 5 = y.$$

Thus $gof(x) = I_x$ and $fog(y) = I_y$.

Where I is identify function.

This implies that f and g are bijections and inverses to each other.

Hence f is a bijection and $f^{-1}(y) = \frac{y+5}{3}$.

Replacing y by x we get, $f^{-1}(x) = \frac{x+5}{3}$

13. The weight of the muscles of a man is a function of his body weight x and can be expressed as W(x) = 0.35x. Determine the domain of this function. **[Qy. - 2023]**

Sol : Given $W(x) = 0.35x$

(Note that x is positive real numbers)

$W(0) = 0$, $W(1) = 0.35$,

$W(2) = 7$, $W(\infty) = \infty$

Since x denotes the number of men, it will take only positive integers.

$\therefore W = W \rightarrow R^+$

Hence the domain is the set of whole numbers. (or) $x > 0$.

14. The distance of an object falling is a function of time t and can be expressed as $s(t) = -16t^2$. Graph the function and determine if it is one-to-one.

Sol : Given $s(t) = -16t^2$

Now, $s(t_1) = s(t_2)$

$\Rightarrow -16t_1^2 = -16t_2^2 \Rightarrow t_1^2 = t_2^2$

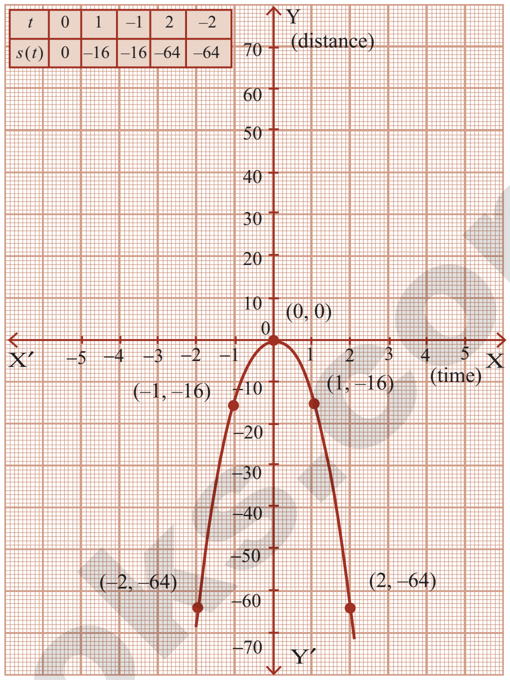
$\Rightarrow \pm t_1 = \pm t_2$

Since $s(t_1) = s(t_2) \neq t_1 = t_2$,

the function $s(t)$ is not one-one.

Graph of $s(t) = -16t^2$

Let X - axis represents the time and Y - axis represents the distance.



15. The total cost of airfare on a given route is comprised of the base cost C and the fuel surcharge S in rupee. Both C and S are functions of the mileage m; C(m) = 0.4 m + 50 and S(m) = 0.03 m. Determine a function for the total cost of a ticket in terms of the mileage and find the airfare for flying 1600 miles.

Sol : Given base cost function and fuel surcharge function are as follows:

$c(m) = 0.4\,m + 50$

and $s(m) = 0.03\,m$.

$\therefore$ Total cost of a ticket $= c(m) + s(m)$

$\therefore f(x) = 0.4\,m + 50 + 0.03\,m$

Total cost $= 0.43\,m + 50$

Given m = 1600 miles

Airfare for flying 1600 miles = 0.43(1600) +50

= ₹738

16. A salesperson whose annual earnings can be represented by the function A(x) = 30,000 + 0.04x, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function S(x) = 25,000 + 0.05x. Find (A + S) (x) and determine the total family income if they each sell ₹1,50,00,000 worth of merchandise.

Sol : Given $A(x) = 30,000 + 0.04\,x$

$S(x) = 25,000 + 0.05\,x$.

$\therefore (A + S)(x) = 30,000 + 0.04x + 25,000 + 0.05x$

$= 55,000 + 0.09x$

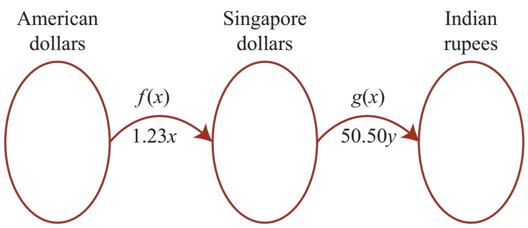
Given x = ₹1,50,00,000

Then Family income is $= 55{,}000 + 0.09\,(1{,}50{,}00{,}000)$
$= 55{,}000 + 13{,}50{,}000$
$= 14{,}05{,}000.$

Hence total family income $=$ ₹ 14,05,000.

17. The function for exchanging American dollars for Singapore Dollar on a given day is $f(x) = 1.23x$, where x represents the number of American dollars. On the same day the function for exchanging Singapore Dollar to Indian Rupee is $g(y) = 50.50y$, where y represents the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee.

Sol : Given $f(x) = 1.23x$ where x represents the number of American dollars and $g(y) = 50.50y$ where y represents the number of Singapore dollars.



To convert American dollars to Indian rupees, we have to find out go $f(x)$

$\therefore$ go $f(x) = g(f(x)) = g(1.23x)$
$= 50.50[1.23x] = 62.115x$

$\therefore$ The function for exchange rate of American dollars in terms of Indian rupee is $gof(x) = 62.115x$.

18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimates that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function $D(x) = 200 - x$. Express his day revenue, total cost and profit on this meal as a function of x.

Sol : Number of customers $= 200 - x$
Cost of one meal $=$ ₹100
Total cost $= 100\,(200 - x)$
Revenue on one meal $= x$
Total revenue $= x\,(200 - x)$
Profit $=$ Revenue – Cost
$=$ ₹ $x(200 - x) - 100\,(200 - x)$

19. The formula for converting from Fahrenheit to Celsius temperatures is $y = \dfrac{5x}{9} - \dfrac{160}{9}$. Find the inverse of this function and determine whether the inverse is also a function.

Sol :

Let $f(x) = \dfrac{5x - 160}{9}$

Given $y = \dfrac{5x}{9} - \dfrac{160}{9} \Rightarrow y = \dfrac{5x - 160}{9}$

Then $9y = 5x - 160$

$\Rightarrow 5x = 9y + 160 \Rightarrow x = \dfrac{9y + 160}{5}$

Let $g(y) = \dfrac{9y + 160}{5}$.

Now $gof(x) = g[f(x)] = g\left(\dfrac{5x - 160}{9}\right)$

$= \dfrac{9\left(\dfrac{5x - 160}{9}\right) + 160}{5}$

$= \dfrac{5x - 160 + 160}{5} = \dfrac{5x}{5} = x$

and $fog(y) = f(g(y)) = f\left(\dfrac{9y + 160}{5}\right)$

$= \dfrac{5\left(\dfrac{9y + 160}{5}\right) - 160}{9} = \dfrac{9y + 160 - 160}{9} = y$

Thus $gof = I_x$ and $fog = I_y$.

This implies that f and g are bijections and inverses to each other.

$\Rightarrow f^{-1}(y) = \dfrac{9y + 160}{5}$

Replacing y by x, we get $f^{-1}(x) = \dfrac{9x + 160}{5} = \dfrac{9x}{5} + 32$

$\therefore$ The inverse function is bijection.

20. A simple cipher takes a number and codes it, using the function $f(x) = 3x - 4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y = x$ (by drawing the lines).

Sol :

Given $f(x) = 3x - 4$

Let $y = 3x - 4 \Rightarrow y + 4 = 3x$

$\Rightarrow x = \dfrac{y + 4}{3}$

Let $g(y) = \dfrac{y + 4}{3}$.

Now $gof(x) = g(f(x)) = g(3x - 4)$

$= \dfrac{3x - 4 + 4}{3} = \dfrac{3x}{3} = x$

and $fog(y) = f(g(y)) = f\left(\dfrac{y + 4}{3}\right)$

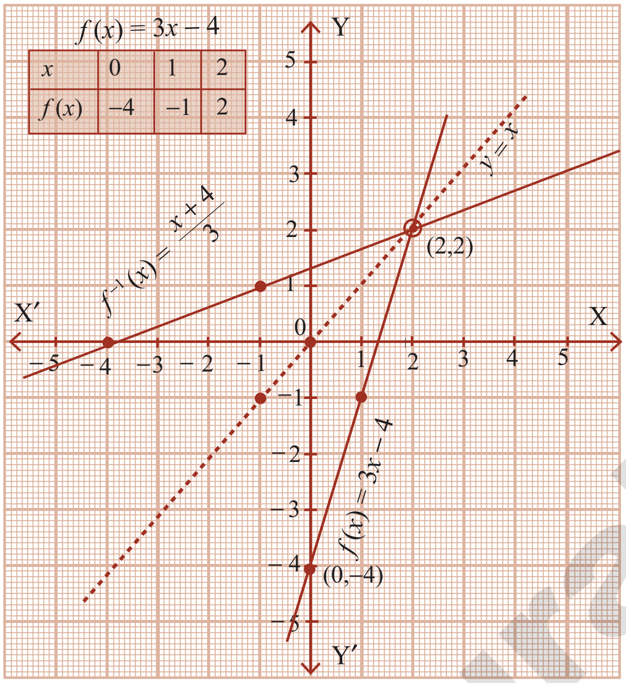
$= 3\left(\dfrac{y + 4}{3}\right) - 4 = y + 4 - 4 = y$

Thus, $gof(x) = I_x$ and $fog(y) = I_y$.

This implies that f and g are bijections and inverses to each other.

Hence f is bijection and $f^{-1}(y) = \dfrac{y+4}{3}$

Replacing y by x, we get $f^{-1}(x) = \boxed{\dfrac{x+4}{3}}$.



Hence, the graph of $y = f^{-1}(x)$ is the reflection of the graph of f in $y = x$

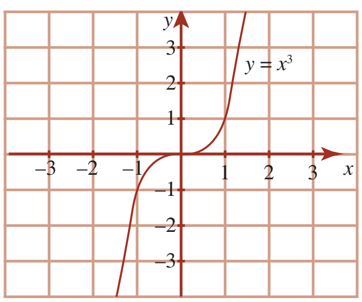
Exercise 1.4

1. For the curve $y = x^3$ given in figure draw,

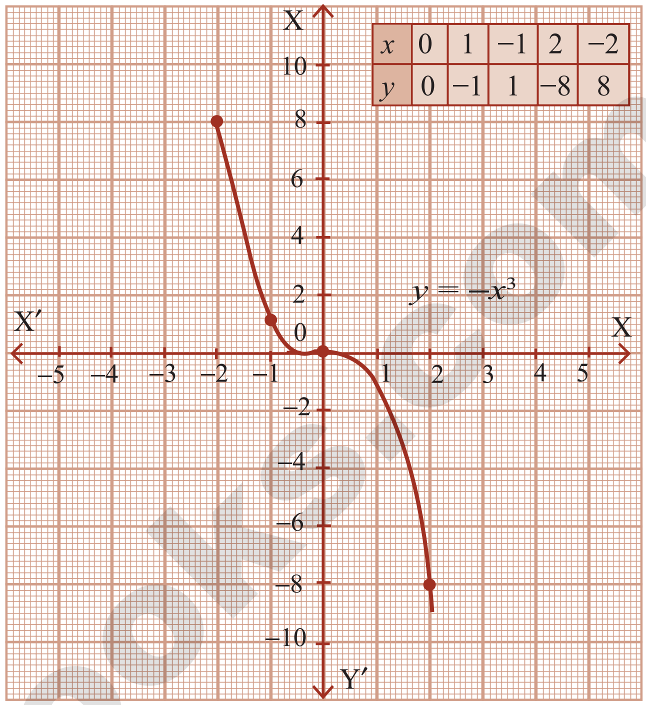
(i) $y = -x^3$ [Qy. - 2019] (ii) $y = x^3 + 1$

(iii) $y = x^3 - 1$ (iv) $y = (x+1)^3$

with the same scale.



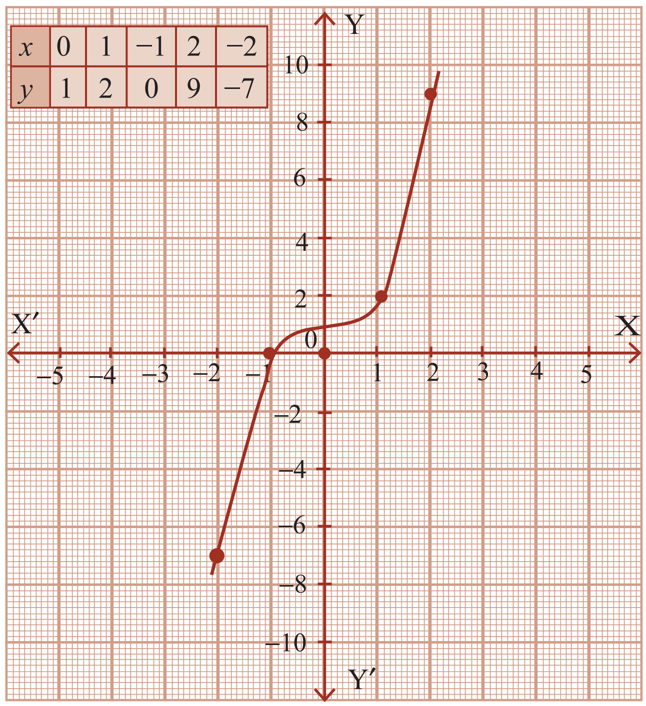
Sol : **(i) $y = -x^3$**



Let $f(x) = x^3$

Since $y = -f(x)$, this is the reflection of the graph of f about the x-axis

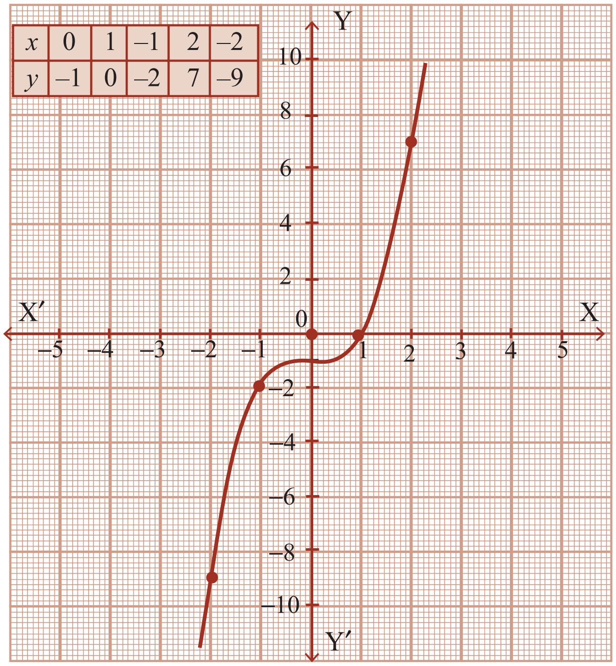
(ii) $y = x^3 + 1$ **[Qy. - 2019]**



Let $f(x) = x^3$

Since $y = f(x) + 1$, this is the graph of $f(x)$ shifts to the upward for one unit

(iii) $y = x^3 - 1$ **[Qy. - 2019]**

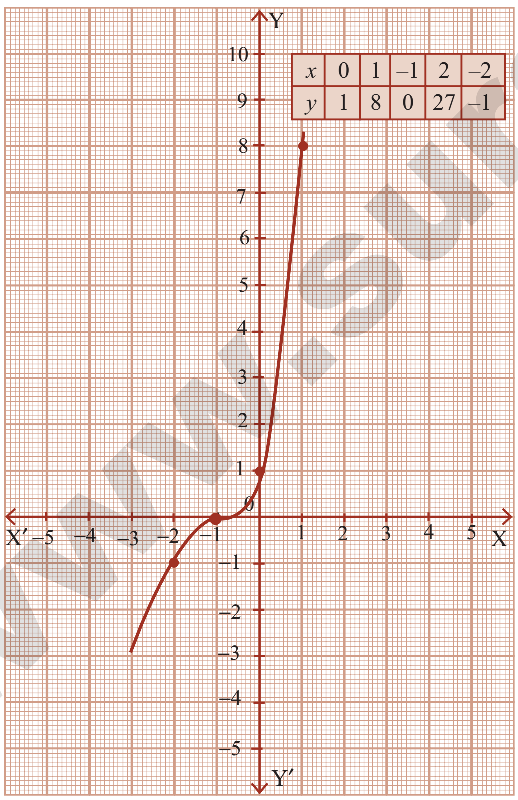


x	0	1	–1	2	–2
y	–1	0	–2	7	–9

Let $f(x) = x^3$

Since $y = f(x) - 1$, this is the graph of $f(x)$ shifts to the downward for one unit.

(iv) $y = (x + 1)^3$ **[Qy. - 2019; Govt. MQP - 2018]**



x	0	1	–1	2	–2
y	1	8	0	27	–1

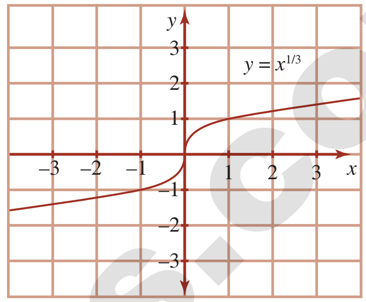
Let $f(x) = x^3$

$y = (x + 1)^3$, causes the graph of $f(x)$ shifts to the left for one unit.

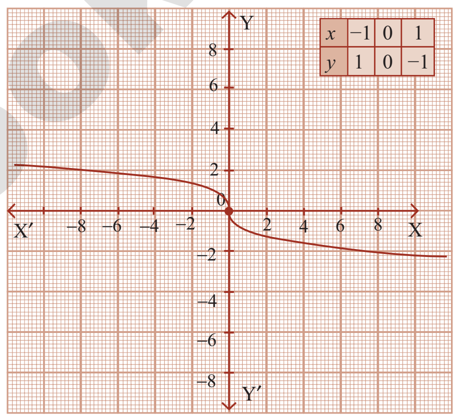
2. **For the curve, $y = x^{\left(\frac{1}{3}\right)}$ given in figure draw.**

(i) $y = -x^{\left(\frac{1}{3}\right)}$ **(ii)** $y = x^{\left(\frac{1}{3}\right)} + 1$

(iii) $y = x^{\left(\frac{1}{3}\right)} - 1$ **(iv)** $(x+1)^{\left(\frac{1}{3}\right)}$



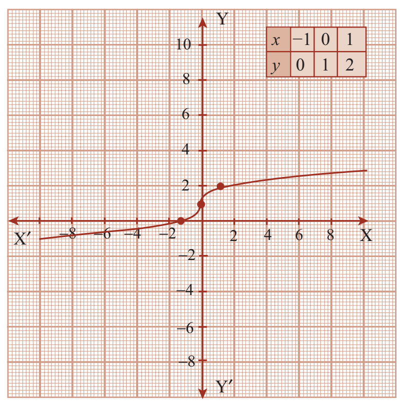
Sol : (i) $y = -x^{\frac{1}{3}}$



x	–1	0	1
y	1	0	–1

Then $y = -x^{\frac{1}{3}}$ is the reflection of the graph of $y = x^{\frac{1}{3}}$ about the x-axis.

(ii) $y = x^{\frac{1}{3}} + 1$.

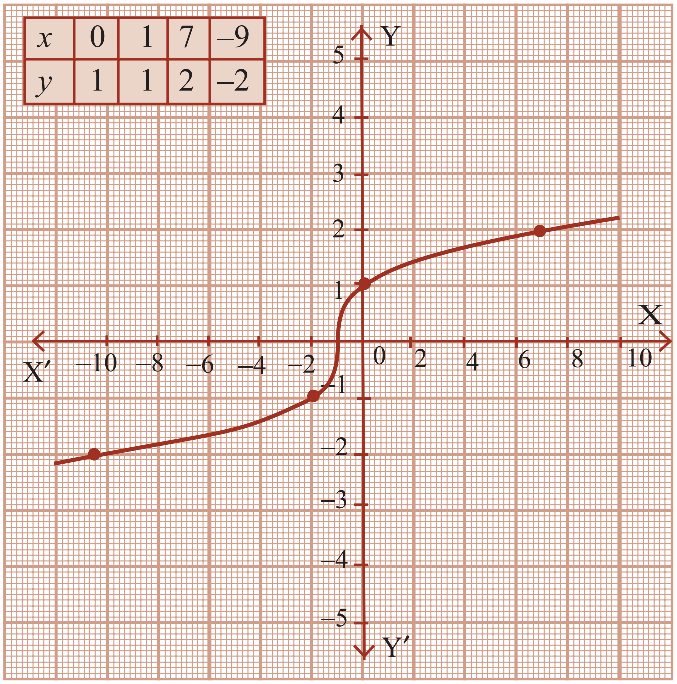


x	–1	0	1
y	0	1	2

Then $y = x^{\frac{1}{3}} + 1$ is the x graph of $y = x^{\frac{1}{3}}$ shifts to the upward for one unit.

(iii) $y = x^{\frac{1}{3}} - 1$.

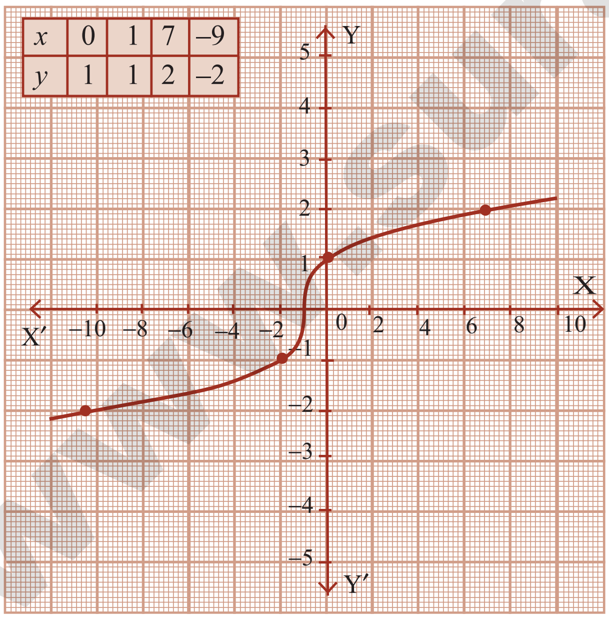
x	0	1	7	–9
y	1	1	2	–2



Then $y = x^{\frac{1}{3}} - 1$ is the graph of $x^{\frac{1}{3}}$ shifts to the downward for one unit.

(iv) $y = (x+1)^{\frac{1}{3}}$

x	0	1	7	–9
y	1	1	2	–2

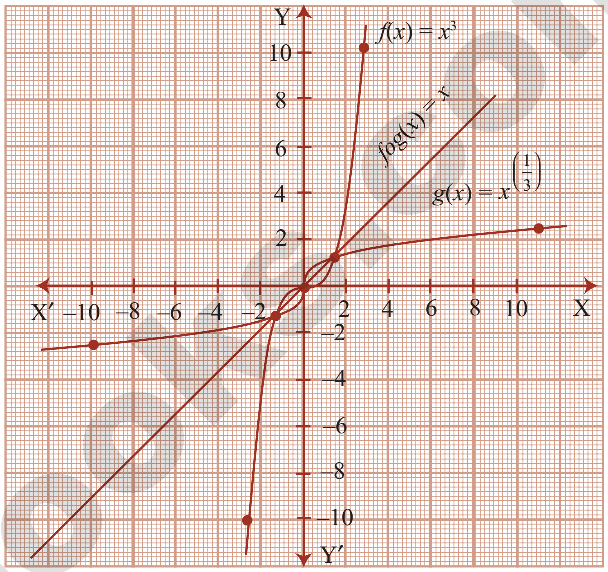


$y = (x+1)^{\frac{1}{3}}$, it causes the graph of $x^{\frac{1}{3}}$, shifts to the left for one unit.

3. Graph the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ on the same co-ordinate plane. Find fog and graph it on the plane as well. Explain your results.

Sol : Given functions are $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$.

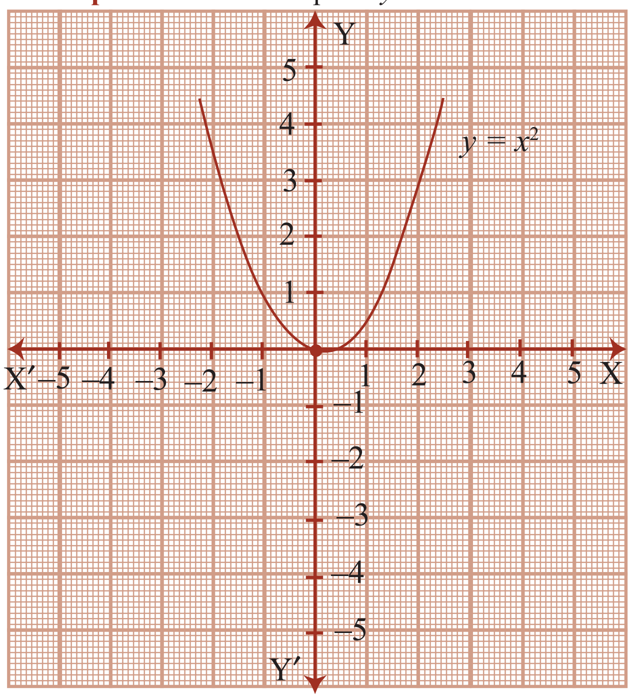
Now, $fog(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = \left(x^3\right)^{\frac{1}{3}} = x$



Since fog$(x) = x$ is symmetric about the line $y = x$, $g(x)$ is the inverse of $f(x)$ $\therefore g(x) = f^{-1}(x)$.

4. Write the steps to obtain the graph of the function $y = 3(x-1)^2 + 5$ from the graph $y = x^2$.

Sol : **Step 1 :** Draw the Graph of $y = x^2$



Step 2 :
The graph of $y = (x-1)^2$, shifts to the right for 1 unit.

Step 3 :
The graph of $y = 3\,(x-1)^2$, compresses towards the Y - axis that is moves away from the X-axis since the multiplying factor is 3 which is greater than 1.

Step 4 :
The graph of $y = 3\,(x-1)^2 + 5$, causes the shift to the upward for 5 units.

5. From the curve $y = \sin x$, graph the functions.

(i) $y = \sin(-x)$ **(ii)** $y = -\sin(-x)$,

(iii) $y = \sin\left(\frac{\pi}{2} + x\right)$ **which is $\cos x$.**

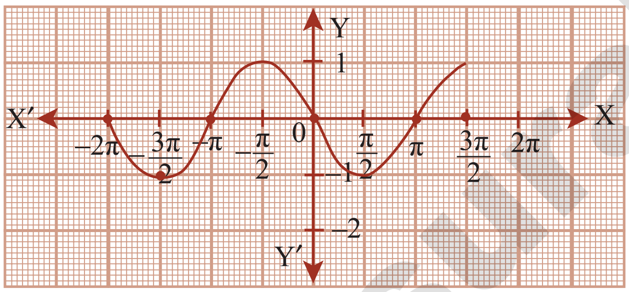
(iv) $y = \sin\left(\frac{\pi}{2} - x\right)$ **which is also $\cos x$.**

(refer trigonometry)

Sol :

(i) $y = \sin(-x)$

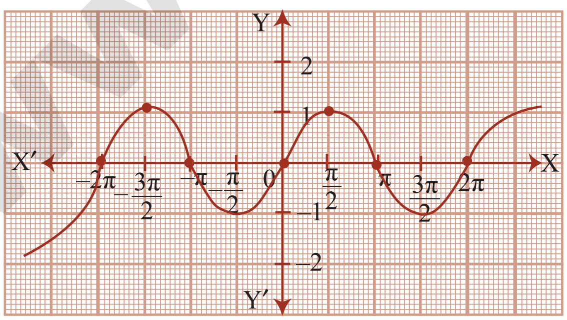
x	-2π	$-3\frac{\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	π	$3\frac{\pi}{2}$
y	0	-1	0	-1	0	1



Then $y = \sin(-x)$ is the reflection of the graph of $\sin x$, about y-axis.

(ii) $y = -\sin(-x)$

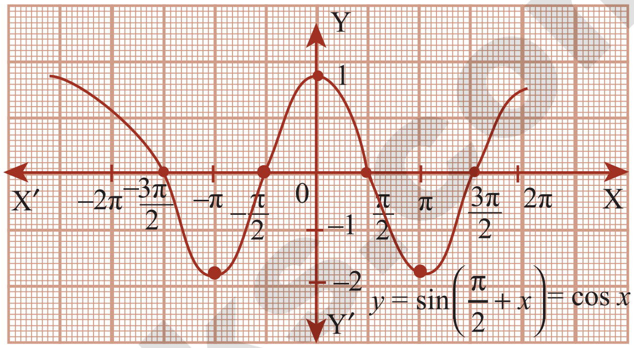
x	-2π	$-3\frac{\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$3\frac{\pi}{2}$	2π
y	0	1	0	-1	0	1	0	-1	0



$y = -\sin(-x)$ is the reflection of $y = \sin(-x)$ which is same as $y = \sin x$.

(iii) $y = \sin\left(\frac{\pi}{2} + x\right)$

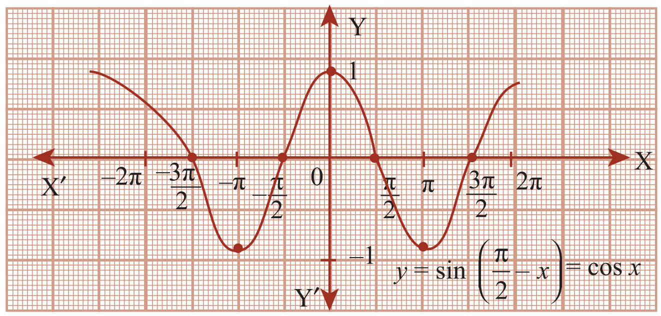
x	-2π	$-3\frac{\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$3\frac{\pi}{2}$	2π
y	1	0	-1	0	1	0	-1	0	1



Then $y = \sin\left(\frac{\pi}{2} + x\right)$ it causes the shift to the left for $\frac{\pi}{2}$ units.

(iv) $y = \sin\left(\frac{\pi}{2} - x\right)$

x	-2π	$-3\frac{\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$3\frac{\pi}{2}$	2π
y	1	0	-1	0	1	0	-1	0	1



Then $y = \sin\left(\frac{\pi}{2} - x\right)$ causes the shift to the right for $\frac{\pi}{2}$ unit to the $\sin(-x)$ curve.

6. From the curve $y = x$, draw

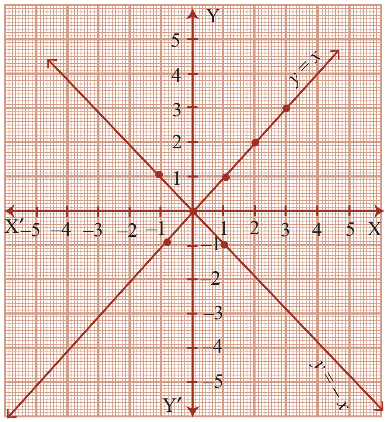
(i) $y = -x$ **(ii)** $y = 2x$

(iii) $y = x + 1$ **(iv)** $y = \frac{1}{2}x + 1$

(v) $2x + y + 3 = 0$.

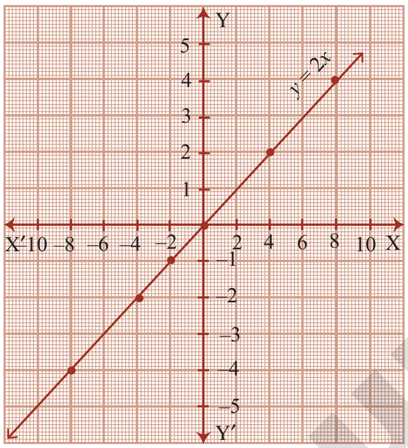
Sol : Graph of $y = x$ and

(i) $y = -x$



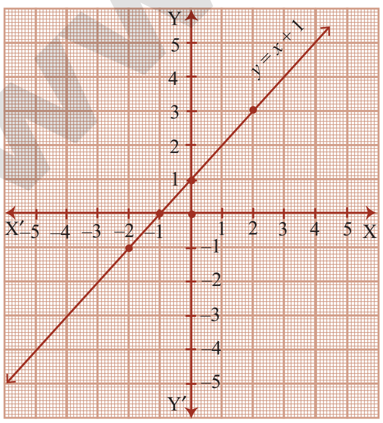
Graph of $y = -x$ is the reflection of the graph of $y = x$ about the X - axis.

(ii) $y = 2x$



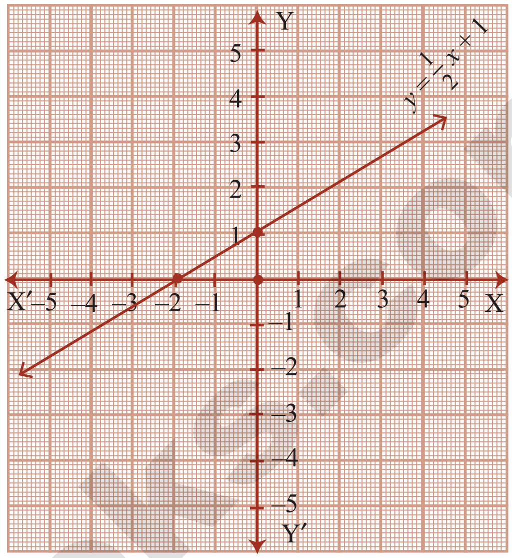
The graph of $y = 2x$ compresses towards the Y-axis that is moves away from the X-axis since the multiplying factor is 2, which is greater than 1.

(iii) $y = x + 1$



The graph of $y = x + 1$, causes the shift to the upward for one unit.

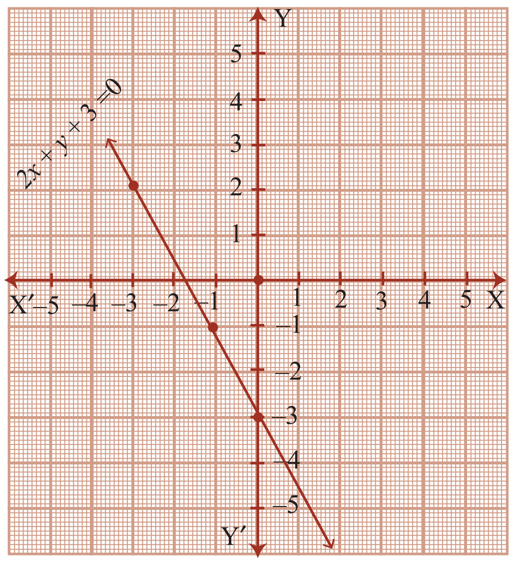
(iv) $y = \frac{1}{2}x + 1$



The graph of $y = \frac{1}{2}x + 1$, stretches towards the X-axis since the multiplying factor is $\frac{1}{2}$ which is less than one and shifts to the upward for one unit.

(v) $2x + y + 3 = 0.$

$\Rightarrow y = -2x - 3$



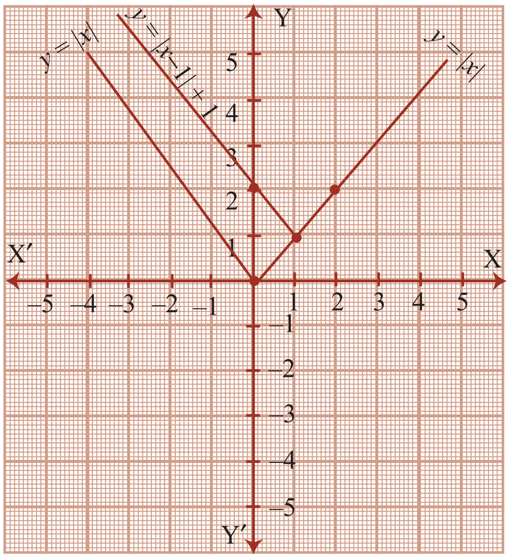
The graph of $y = -2x - 3$, stretches towards the X-axis since the multiplying factor is -2 which is less than one and causes the shifts to the downward for 3 units.

7. From the curve $y = |x|$, draw

(i) $y = |x - 1| + 1$ **(ii)** $y = |x + 1| - 1$

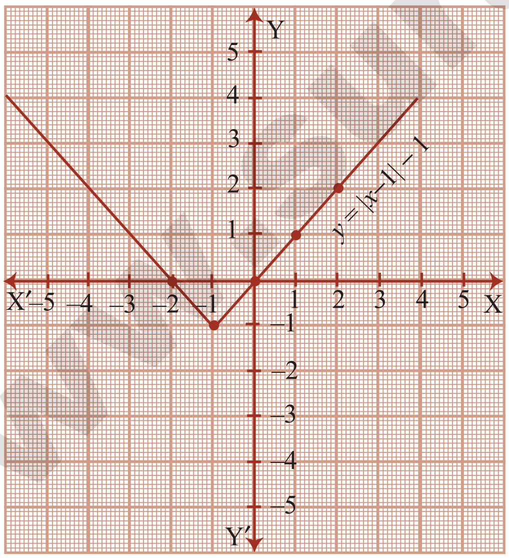
(iii) $y = |x + 2| - 3$.

Sol : **(i)** $y = |x - 1| + 1$



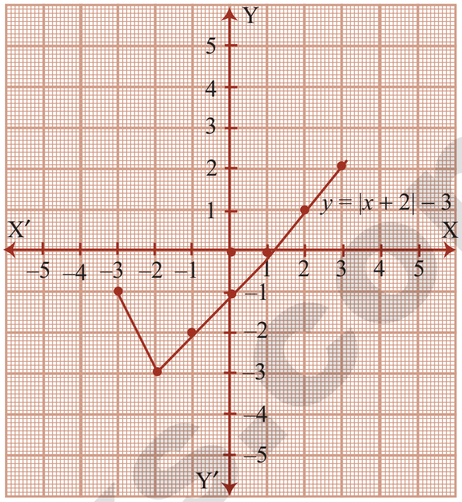
The graph of $y = |x - 1| + 1$, shifts to the right for one unit and causes the shift to the upward for one unit.

(ii) $y = |x + 1| - 1$



The graph of $y = |x + 1| - 1$, shifts to the left for one unit and causes the shift to the downward for one unit.

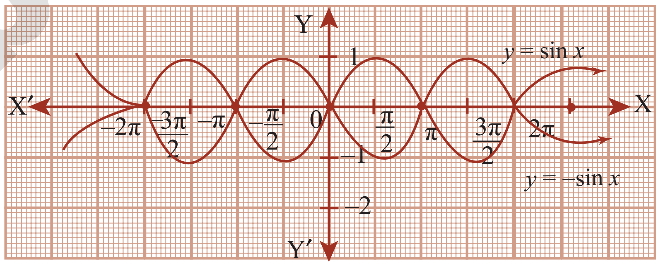
(iii) $y = |x + 2| - 3$



The graph of $y = |x + 2| - 3$, shifts to the left for 2 units and causes the shift to the downward for 3 units.

8. From the curve $y = \sin x$, draw $y = \sin |x|$ (Hint: $\sin(-x) = -\sin x$.) **[June - 2019]**

Sol : $y = \sin |x|$



$$\text{We know } |x| = \begin{cases} x \text{ if } & x \geq 0 \\ -x \text{ if } & x < 0 \end{cases}$$

$$\therefore \sin |x| = \sin x \text{ if } x \geq 0$$

$$\text{and } \sin |x| = \sin(-x) = -\sin x \text{ if } x < 0.$$

The graph of $y = \sin(-x) = -\sin x$ is the reflection of the graph of $\sin x$ about Y - axis.

Exercise 1.5

Choose the correct or the most suitable answer.

1. If $A = \{(x, y) : y = e^x, x \in \mathbb{R}\}$ and $B = \{(x, y) : y = e^{-x}, x \in \mathbb{R}\}$ then $n(A \cap B)$ is **[First Mid - 2018; Hy. - 2019]**

(1) Infinity (2) 0 (3) 1 (4) 2

[Ans : (3) 1**]**

Hint : A = {(0,1) (1, e) (2, e^2)...}
B = {(0,1) (1, e^{-1}) (2, e^{-2})...}
$\Rightarrow n = (A \cap B) = (0, 1) = 1$

2. If A = {$(x, y) : y = \sin x,\ x \in \mathbb{R}$} and B = {$(x, y) : y = \cos x,\ x \in \mathbb{R}$} then A ∩ B contains
[Govt. MQP - 2018; Mar. - 2020; Aug. - 2022; QY-'24]

(1) no element
(2) infinitely many elements
(3) only one element
(4) cannot be determined.

[Ans : (2) infinitely many elements**]**

3. The relation R defined on a set A = {0,–1, 1, 2} by xRy if $|x^2 + y^2| \le 2$, then which one of the following is true? **[Hy. - 2023]**

(1) R = {(0, 0), (0,–1), (0, 1), (–1, 0), (–1, 1), (1, 2), (1, 0)}
(2) R^{-1} = {(0, 0), (0,–1), (0, 1), (–1, 0), (1, 0)}
(3) Domain of R is {0,–1, 1, 2}
(4) Range of R is {0,–1, 1}

[Ans : (4) Range of R is {0, –1, 1}**]**

Hint : R = $|x^2 + y^2| \le 2$ = {(0,0) (0, –1) (0,1) (–1,0) (–1,–1) (–1,1) (1,0) (1,–1) (1,1) (2,0)}
∴ Range ={0,–1, 1}

4. If $f(x) = |x - 2| + |x + 2|,\ x \in \mathbb{R}$, then

(1) $f(x) = \begin{cases} -2x & \text{if } x \in (-\infty, -2] \\ 4 & \text{if } x \in (-2, 2] \\ 2x & \text{if } x \in (2, \infty) \end{cases}$

(2) $f(x) = \begin{cases} 2x & \text{if } x \in (-\infty, -2] \\ 4x & \text{if } x \in (-2, 2] \\ -2x & \text{if } x \in (2, \infty) \end{cases}$

(3) $f(x) = \begin{cases} -2x & \text{if } x \in (-\infty, -2] \\ -4x & \text{if } x \in (-2, 2] \\ 2x & \text{if } x \in (2, \infty) \end{cases}$

(4) $f(x) = \begin{cases} -2x & \text{if } x \in (-\infty, -2] \\ 2x & \text{if } x \in (-2, 2] \\ 2x & \text{if } x \in (2, \infty) \end{cases}$

[Ans : (1) $f(x) = \begin{cases} -2x & \text{if } x \in (-\infty, -2] \\ 4 & \text{if } x \in (-2, 2] \\ 2x & \text{if } x \in (2, \infty) \end{cases}$**]**

Hint : Let $x \in (-\infty, -2)$,
let x = –3 then
$f(x) = |-5| + |1| = 6 = -2x$
$x \in (-2, 2)$, let x = 0 then
$f(x) = |0 - 2| + |0 + 2| = 4$
$x \in (2, \infty)$, let x = 4 then
$f(x) = |2| + |6| = 8 = 2x$

5. Let ℝ be the set of all real numbers. Consider the following subsets of the plane ℝ × ℝ: S = {$(x, y) : y = x + 1$ and $0 < x < 2$} and T = {$(x, y) : x - y$ is an integer}. Then which of the following is true?

(1) T is an equivalence relation but S is not an equivalence relation.
(2) Neither S nor T is an equivalence relation
(3) Both S and T are equivalence relation
(4) S is an equivalence relation but T is not an equivalence relation.

Hint : $x - y$ is an integer $\Rightarrow x$Ry
(i) $x - x = 0$ is an integer. ∴ xRx reflexive
(ii) $(x - y)$ is an integer $\Rightarrow y - x$ is also an integer $\Rightarrow$ symmetric
(iii) If $(x - y)$ is an integer $\Rightarrow y - z$ is an integer by adding $x - z$ is also an integer. $\Rightarrow$ Transitive
∴ T is equivalence.
(iv) $y = x + 1 \Rightarrow x$Rx is not true. S is not an equivalence relation.
∴ T is an equivalence relation but S is not.

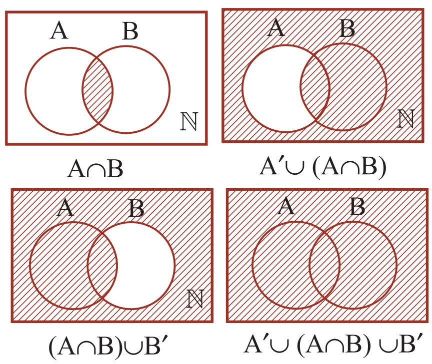
[Ans : (1) T is an equivalence relation but S is not an equivalence relation**]**

6. Let A and B be subsets of the universal set ℕ, the set of natural numbers. Then A′∪[(A∩B)∪B′] is
[First Mid - 2018; May - 2022; QY-'23 & '24]

(1) A (2) A′ (3) B (4) ℕ

[Ans : (4) ℕ**]**

Hint :



7. The number of students who take both the subjects Mathematics and Chemistry is 70. This represents 10% of the enrollment in Mathematics and 14% of the enrollment in Chemistry. The number of students take at least one of these two subjects, is

(1) 1120 (2) 1130
(3) 1100 (4) insufficient data

[Ans : (2) 1130**]**

Hint : $M \cap C = 70$

Which is 10% of M and 14% of C

$M = 700$

$C = 500$

$M \cup C = 700 + 500 - 70 = 1130$

8. If $n((A \times B) \cap (A \times C)) = 8$ and $n(B \cap C) = 2$, then $n(A)$ is [CRT - 2022; July - 2023; Mar. - 2024]

(1) 6 (2) 4 (3) 8 (4) 16

[Ans : (2) 4]

Hint : $(A \times B) \cap (A \times C) = A \times (B \cap C)$

$n[(A \times B) \cap (A \times C)] = 8$

$n(B \cap C) = 2$

$n(A) = 4$

9. If $n(A) = 2$ and $n(B \cup C) = 3$, then $n[(A \times B) \cup (A \times C)]$ is [Qy. - 2018 & 2023]

(1) 2^3 (2) 3^2 (3) 6 (4) 5

Hint : $n[(A \times B) \cup (A \times C)\} = n(A) \times n(B \cup C)$

$= 2 \times 3 = 6$ **[Ans : (3) 6]**

10. If two sets A and B have 17 elements in common, then the number of elements common to the set A × B and B × A is [CRT - 2022]

(1) 2^{17} (2) 17^2

(3) 34 (4) insufficient data

[Ans : (2) 17^2]

Hint : Let $A = \{1, 2, 3, 4\}$

$B = \{5, 2, 3, 6\}$

A and B have two elements in common

Number of elements common to A × B

and $B \times A = 2 \times 2 = 2^2$

Similarly here we have 17^2 elements common

11. For non-empty sets A and B, if $A \subset B$ then $(A \times B) \cap (B \times A)$ is equal to [Hy. - 2018]

(1) $A \cap B$ (2) $A \times A$

(3) $B \times B$ (4) none of these.

Hint : Let A= (a, b) B = (a, b, c) **[Ans : (2) A × A]**

$A \times B = \{(a, a), (a, b) (a, c) (b, a) (b,b) (b, c)\}$

$B \times A = \{(a, a), (a, b) (b, a) (b, b) (c,a) (c, b)\}$

$(A \times B) \cap (B \times A) = \{(a, a), (a, b) (b, a) (b,b)\}$

$= A \times A$

12. The number of relations on a set containing 3 elements is [Govt.MQP & First Mid-2018; CRT - 2022; April - 2023; July-'24]

(1) 9 (2) 81 (3) 512 (4) 1024

[Ans : (3) 512]

Hint : Let $S = \{a, b, c\}$

$n(S) = 3 \Rightarrow n(S \times S) = 9$

Number of relations is $n\{P(S \times S)\} = 2^9 = 512$

13. Let R be the universal relation on a set X with more than one element. Then R is

(1) not reflexive (2) not symmetric

(3) transitive (4) none of the above

[Ans : (3) transitive]

Hint : Let $X = \{a, b, c\}$

Then R = Universal relation

$= \{(a, a), (a, b) (a, c) (b, a) (b,b) (b, c) (c,a) (c, b)(c, c)\}$

It is transitive

14. Let X = {1, 2, 3, 4} and R = {(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (2, 1), (3, 1), (1, 4),(4, 1)}. Then R is [First Mid - 2018; Sep. 2021]

(1) reflexive (2) symmetric

(3) transitive (4) equivalence

[Ans : (2) symmetric]

Hint : $(4,4) \in R$ not reflexive

Symmetric can be easily checked

$\Rightarrow$ if aRb then bRc.

15. The range of the function $\dfrac{1}{1-2\sin x}$ is [June & Qy. - 2019; CRT - 2022]

(1) $(-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$ (2) $\left(-1, \frac{1}{3}\right)$

(3) $\left[-1, \frac{1}{3}\right]$ (4) $(-\infty, -1] \cup \left[\frac{1}{3}, \infty\right)$

[Ans : (4)$(-\infty, -1] \cup \left[\frac{1}{3}, \infty\right)$]

Hint : $-1 \le \sin x \le 1$

$-2 \le 2 \sin x \le 2$

$2 \ge -2 \sin x \ge -2$ (or) $-2 \le -2 \sin x \le 2$

Adding, 1, $1 - 2 \le 1 - 2 \sin x \le 1 + 2$

$-1 \le 1 - 2 \sin x \le 3$

$-1 \ge \dfrac{1}{1-2\sin x} \ge \dfrac{1}{3}$

$\dfrac{1}{3} \le \dfrac{1}{1-2\sin x} \le -1$ Range is $(-\infty, -1] \cup \left[\frac{1}{3}, \infty\right)$

16. The range of the function $f(x) = |\lfloor x \rfloor - x|, x \in \mathbb{R}$ is [Hy. - 2019; Qy. - 2023]

(1) [0, 1] (2) $[0,\infty)$ (3) [0,1) (4) (0,1)

[Ans : (3) [0,1)]

Hint : $f(x) = |\lfloor x \rfloor - x|$

$f(x) = \lfloor x \rfloor - x$

$f(0) = 0 - 0 = 0$

$f(6.5) = 6 - 6.5 = |-.5| = .5$

$f(-7.2) = 8 - 7.2 = .8$

Range is [0,1)

17. The rule $f(x) = x^2$ is a bijection if the domain and the co-domain are given by **[April - '23; QY-'24]**

(1) $\mathbb{R}, \mathbb{R}$ (2) $\mathbb{R}, (0, \infty)$
(3) $(0,\infty), \mathbb{R}$ (4) $[0,\infty), [0,\infty)$

Hint : The domain is $(0,\infty)$

The codomain is also $(0,\infty)$ **[Ans :** (4) $[0,\infty), [0,\infty)$]

18. The number of constant functions from a set containing *m* elements to a set containing *n* elements is **[Hy. - 2019; July-'24]**

(1) mn (2) m (3) n (4) $m+n$

Hint : By definition it follows **[Ans :** (3) n]

19. The function $f : [0, 2\pi] \to [-1, 1]$ defined by $f(x) = \sin x$ is

[Mar. - 2020; Govt. MQP - 2018; Qy. - 2019 & 2023]

(1) one-to-one (2) onto
(3) bijection (4) cannot be defined

Hint : It is onto not one-one

$$\text{since } \sin 30^\circ = \frac{1}{2}$$

$$\sin 150^\circ = \frac{1}{2}$$

[Ans : (2) onto]

20. If the function $f : [-3, 3] \to S$ defined by $f(x) = x^2$ is onto, then S is **[Mar.- 2020 & 2024; June - 2019]**

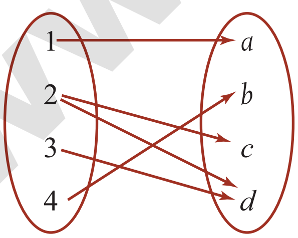
(1) $[-9, 9]$ (2) $\mathbb{R}$ (3) $[-3, 3]$ (4) $[0, 9]$

Hint : $f(0) = 0, f(-3) = 9$ and $f(3) = 9$ **[Ans :** (4) $[0, 9]$]

21. Let X = {1, 2, 3, 4}, Y = {*a*, *b*, *c*, *d*} and *f* = {(1, *a*), (4, *b*), (2, *c*), (3, *d*), (2, *d*)}. Then *f* is

(1) an one-to-one function
(2) an onto function
(3) a function which is not one-to-one
(4) not a function **[Ans :** (4) not a function]

Hint : It is not a function since it has two images.



22. The inverse of $f(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 & \text{if } 1 \le x \le 4 \\ 8\sqrt{x} & \text{if } x > 4 \end{cases}$ **is**

(1) $f^{-1}(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \le x \le 16 \\ \dfrac{x^2}{64} & \text{if } x > 16 \end{cases}$

(2) $f^{-1}(x) = \begin{cases} -x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \le x \le 16 \\ \dfrac{x^2}{64} & \text{if } x > 16 \end{cases}$

(3) $f^{-1}(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \le x \le 16 \\ \dfrac{x^2}{64} & \text{if } x > 16 \end{cases}$

(4) $f^{-1}(x) = \begin{cases} 2x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \le x \le 16 \\ \dfrac{x^2}{8} & \text{if } x > 16 \end{cases}$

[Ans : (1) $f^{-1}(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \le x \le 16 \\ \dfrac{x^2}{64} & \text{if } x > 16 \end{cases}$]

Hint :

Let $y = x$ then $x = y \Rightarrow f^{-1}(x) = x$

Let $y = x^2$ then

$y = \sqrt{x} \Rightarrow f^{-1}(x) = \sqrt{x}$

Let $y = 8\sqrt{x}$ then $\frac{y^2}{64} = x \Rightarrow f^{-1}(x) = \frac{x^2}{64}$

23. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 - |x|$. Then the range of *f* is **[Govt. MQP; Qy. & Hy. - 2018; Hy. - 2023]**

(1) $\mathbb{R}$ (2) $(1,\infty)$ (3) $(-1,\infty)$ (4) $(-\infty, 1]$

[Ans : (4) $(-\infty, 1]$]

Hint : $f : R \to R$ defined by

$$f(x) = 1 - |x|$$

The range is $(-\infty, 1]$, $f(-\infty) = -\infty$

$$f(0) = 1$$

$$f(\infty) = -\infty$$

24. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sin x + \cos x$ is

(1) an odd function
(2) neither an odd function nor an even function
(3) an even function
(4) both odd function and even function.

Hint :
$$f(x) = \sin x + \cos x$$
$$f(-x) = \sin(-x) + \cos(-x) = -\sin x + \cos x$$
$$-f(-x) = \sin x - \cos x$$

$f(x)$ is neither odd function nor even function.

[Ans : (2) neither an odd function nor an even function]

25. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \dfrac{(x^2+\cos x)(1+x^4)}{(x-\sin x)(2x-x^3)} + e^{-|x|}$ is

(1) an odd function
(2) neither an odd function nor an even function
(3) an even function
(4) both odd function and even function.

[Ans : (3) an even function]

Hint :
$$f(x) = \frac{(x^2+\cos x)}{(x-\sin x)(2x-x^3)} + e^{-|x|}$$
$$f(-x) = \frac{(-x^2)+\cos(-x)}{[-x-\sin(-x)][-2x-(-x)^3]} + e^{-|x|}$$
$$= \frac{x^2+\cos x}{(x-\sin x)(2x-x^3)} + e^{-|x|} = f(x)$$

Here $f(x)$ is even function.

GOVERNMENT EXAM QUESTIONS

SECTION - A (1 MARK)

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER.

1. If A = {1, 2, 3}, B = {1, 4, 6, 9} and R is a relation from A to B defined by "x is greater than y". The range of R is **[Sep. - 2020]**

(1) {1, 4, 6, 9} (2) {4, 6, 9}
(3) {1} (4) none of these

Hint : {(2, 1), (3, 1)} **[Ans : (3) {1}]**

2. If n(A) = 5 and n (B) = 7 then the number of subsets of A × B is **[Qy. - 2019]**

(1) 2^{35} (2) 2^{49} (3) 2^{25} (4) 2^{70}

[Ans : (1) 2^{35}]

Hint : $n(A \times B) = n(A) \times n(B) = 5 \times 7 = 35$

Number of subsets of A × B = 2^{35}

3. The relation "less than" in the set of natural number is **[Qy. - 2019]**

(1) only symmetric (2) only transitive
(3) only reflexive (4) Equivalence

Hint : Let $m, n, 0 \in \mathbb{N}$ **[Ans : (2) only transitive]**

(i) Reflexive
$(m, m) = m < m$, which is not true.
Not Reflexive.

(ii) Symmetric
$(m, n) = m < n$ ... (i)
$(n, m) = n < m$ ... (ii)
This is not possible

(iii) Transitive
$(m, n) = m < n$ ... (i)
$(n, 0) = n < 0$ ... (ii)
$\therefore m < m < 0 \Rightarrow m < 0$ $(m, 0)$ exist
It is transitive

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^4$. Choose the correct answer. **[Qy. - 2018]**

(1) f is one - one onto (2) f is onto
(3) f is one - one but not onto
(4) f is neither one - one nor onto

[Ans : (4) f is neither one - one nor onto]

Hint :
$$f(2) = 2^4 = 16$$
$$f(-2) = (-2)^4 = 16.$$

Therefore two elements in the domain have same image and hence co domain ≠ Range. **[Ans : (3) x]**

5. Let $f : \mathbb{R} \to \mathbb{R}$ to given by $f(x) = (3-x^3)^{\frac{1}{3}}$. then $fof(x)$ is **[Qy. - 2018]**

(1) $x^{\frac{1}{a}}$ (2) x^a (3) x (4) $3 - x^a$

Hint : **[Ans : (3) x]**
$$fof(x) = f\left[(3-x^3)^{\frac{1}{3}}\right]$$
$$= \left(3-\left[3-x^3\right]\right)^{\frac{1}{3}} = x$$

6. Let A = {–2, –1, 0, 1, 2} and f : A → Z be given by $f(x) = x^2 - 2x - 3$ then preimage of 5 is **[First Mid - 2018]**

(1) –2 (2) –1 (3) 0 (4) 1

[Ans : (1) – 2]

Hint : $f(-2) = (-2)^2 - 2(-2) - 3 = 4 + 4 - 3 = 5$

7. If A = {(x, y)/y = e^x, $x \in [0, \infty)$}and B = {(x, y)/y = sin x, $x \in [0, \infty)$}then $n(A \cap B)$ is [Mar. - 2019]

(1) ∞ (2) 1 (3) φ (4) 0

Hint : $n(A \cap B) = 0$ **[Ans : (4) 0]**

8. If : $\mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x| - 5$, then the range of f is: [Mar. - 2019]

(1) $(-\infty, -5)$ (2) $(-\infty, 5)$
(3) $[-5, \infty)$ (4) $(-5, \infty)$

Hint : $0 \le |x| < \infty, x \in \mathbb{R}$ **[Ans : (3) $[-5, \infty)$]**

$0 - 5 \le |x| - 5 < \infty$; $-5 \le |x| - 5 < \infty$

9. Given $n(A) = 7$, $n(B) = 8$ and $n(A \cup B) = 10$ find $n[P(A \cap B)]$ [CRT - 2022]

(1) 30 (2) 16 (3) 32 (4) 64

[Ans : (3) 32]

Hint : $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow n(A \cap B) = 5$

$n[P(A \cap B)] = 2^5 = 32$

10. The number of reflexive relations on a set containing n elements is : [July - 2023]

(1) $2^{\frac{(n^2+n)}{2}}$ (2) 2^{n^2-n} (3) 2^n (4) 2^{-n}

[Ans : (2) 2^{n^2-n}]

11. The value $16^{\frac{3}{4}}$ = ... [Qy. - 2023]

(1) 8 (2) $\frac{1}{8}$ (3) 4 (4) $\frac{1}{(4)^3}$

[Ans : (1) 8]

SECTION - B (2 MARKS)

1. If $p(A)$ denotes the power set of A, then find $n(p(p(p(Q))))$. [Qy. - 2019; Sep. - 2021]

Sol : Since $P(\varnothing)$ contains 1 element, $P(P(\varnothing))$ contains 2^1 elements and hence $P(P(P(\varnothing)))$ contains 2^2 elements. That is, 4 elements.

2. If $f(x) = y = \frac{ax - b}{ax - a}$, then prove that $f(y) = x$. [Qy. - 2019]

Sol :

$$y = \frac{ax-b}{cx-a}$$
$$y = ax - b$$
$$ycx - ay = ax - b$$
$$cxy - ax = ay - b$$
$$x(cy - a) = ay - b$$
$$x = \frac{ay-b}{cy-a} = f(y)$$

3. In the set Z of integers, define mRn if $m - n$ is a multiple of 12. Prove that R is an equivalence relation. [Mar. -2020; Govt. MQP & Qy. - 2018; Hy. - 2019]

Sol : As $m - m = 0$ and $0 = 0 \times 12$, hence mRm proving that R is reflexive.

Let mRn. Then $m - n = 12k$ for some integer k; thus

$n - m = 12(-k)$ and hence nRm.

This shows that R is symmetric.

Let mRn and nRp; then

$m - n = 12k$

and $n - p = 12l$ for some integers k and l.

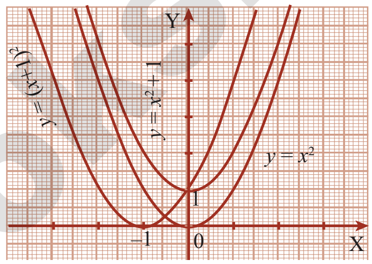
So $m - p = 12(k + l)$ and hence mRp.

This shows that R is transitive.

Thus R is an equivalence relation.

4. Draw the curves of (i) $y = x^2 + 1$ (ii) $y = (x + 1)^2$ by using the graph of curve $y = x$. [Hy. - 2018]

Sol :



$f(x) = x^2 + 1$ causes the graph of the function $f(x) = x^2$ shifts to the upward for one unit.

$f(x) = (x + 1)^2$ causes the graph of the function $f(x) = x^2$ shifts to the left for one unit.

5. Find the number of subsets of A if [May - 2022; July-'24]

A = {X : X = 4n + 1, $2 \le n \le 5$, $n \in \mathbb{N}$} [First Mid - 2018]

Sol : Clearly A = $\{x : x = 4n + 1, n = 2, 3, 4, 5\}$

$n = 2 \Rightarrow 4(2) + 1 = 8 + 1 = 9$

$n = 3 \Rightarrow 4(3) + 1 = 12 + 1 = 13$

$n = 4 \Rightarrow 4(4) + 1 = 16 + 1 = 17$

$n = 5 \Rightarrow 4(5) + 1 = 20 + 1 = 21$

A = $\{x : x = 9, 13, 17, 21\}$

Hence $n(A)$ = 4. This implies that

$n(P(A)) = 2^4 = 16$.

6. Let f = {(1, 4) (2, 5) (3, 5)} and g = {(4, 1) (5, 2) (6, 4)} find gof. Can you find fog? [First Mid - 2018]

Sol : Clearly, gof = {(1, 1), (2, 2), (3, 2)}

But fog is not defined because the range of

g = {1, 2, 4} is not contained in the domain of f = {1, 2, 3}.

7. Define one to one function. [First Mid - 2018]

Sol : A function is said to be one-to-one if each element of the range is associated with exactly one element of the domain. i.e. two different elements in the domain(A) have different images in the co-domain(B).

8. If A = {1, 2, 3, 4} and B = {3, 4, 5, 6}, find $n((A \cup B) \times (A \cap B) \times (A \Delta B))$. **[Govt. MQP- 2018]**

Sol : We have $n(A \cup B) = 6$, **[Sep. - 2020; April - 2023]**

$n(A \cap B) = 2$ and

$n(A \Delta B) = 4$

So, $n((A \cup B) \times n(A \cap B) \times (A \Delta B))$

$= n(A \cup B) \times n(A \cap B) \times n(A \Delta B) = 6 \times 2 \times 4 = 48$

9. If A × A has 9 elements, S = {(a, b) ∈ A × A : a > b}; (2, –1) and (2, 1) are two elements, then find the remaining elements of S. **[Govt. MQP- 2018]**

Sol : $n(A \times A) = 9 \Rightarrow n(A) = 3$

$S = \{(a,b) \in A \times A : a > b\}$

$A = \{-1, 1, 2\}$

$A \times A = \{(-1, -1), (-1, 1), (-1, 2), (1, -1), (1, 1), (1, 2), (2, -1), (2, 1), (2, 2)\}$

$\therefore S = \{(1, -1), (2, -1), (2, 1)\}$

10. Write the use of horizontal line test. **[Mar.- 2019]**

Sol : Variations of the horizontal line test can be used to determine whether a function is surjcetive or bijective.

11. Is it correct to say A × A = {(a, a) : a ∈ A} ? justify your answer. **[Mar. - 2019]**

Sol : No, $A \times A = \{(a, b) : a, b \in A\}$ is only true

Ex: $A = \{1, 2\}$

$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)$

12. If $f : [-2, 2] \to B$ is given by $f(x) = 2x^3$, then find B so that f is onto. **[Hy. - 2019]**

Sol : The minimum value is $f(-2)$ and its maximum value is $f(2)$ which are equal to -16 and 16 respectively. So B is $[-16, 16]$.

13. Let f and g be the two functions from R to R defined by $f(x) = 3x - 4$ and $g(x) = x^2 + 3$ Find fog. **[CRT - 2022; July-2023]**

Sol : We have, $(g \, o \, f)(x) = g(f(x)) = g(3x - 4) = (3x - 4)^2 + 3 = 9x^2 - 24x + 19$

$(f \, o \, g)(x) = f(g(x)) = f(x^2 + 3) = 3(x^2 + 3) - 4 = 3x^2 + 5.$

2. Write the following set in roster form, {The set of positive roots of the equation $(x^2-4)(x^3-27)=0$}.

Sol : $\{(x^2-4)(x^3-27)\}$ **[Hy. - 2023]**

$x^2 - 4 = 0$	$x^3 = 27$
$x^2 = 4$	$x^3 = 3^3$
$x = \pm 2$	$x = 3$
$\boxed{x = 2}$	$\boxed{x = 3}$

$\therefore A = \{2, 3\}$

3. If $n(A) = 10$ and $n(A \cap B) = 3$, find $n((A \cap B)' \cap A)$. **[QY-'24]**

Sol : Refer Text Book Example 1.7

SECTION - C (3 MARKS)

1. Find the range of $f(x) = \dfrac{1}{1 - 3\cos x}$. **[Mar. - 2020; Qy. - 2023]**

Sol : $-1 \le \cos x \le 1$

$\Rightarrow \quad 3 \ge -3\cos x \ge -3$

$\Rightarrow \quad -3 \le -3\cos x \le 3$

$\Rightarrow \quad 1 - 3 \le 1 - 3\cos x \le 1 + 3$

Thus we get $-2 \le 1 - 3\cos x$ and $1 - 3\cos x \le 4$.

By taking reciprocals, we get $\dfrac{1}{1-3\cos x} \le -\dfrac{1}{2}$ and $\dfrac{1}{1-3\cos x} \ge \dfrac{1}{4}$.

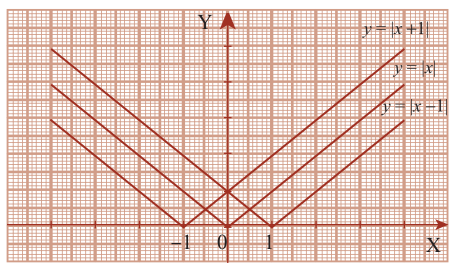
Hence the range of f is $\left(-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{4}, \infty\right)$

2. Find the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$ **[Qy. - 2019; CRT - 2022]**

Sol : The function is defined for all $x \in \mathbb{R}$ except $1 - 2\cos x = 0$. That is, except $\cos x = \dfrac{1}{2}$. That is except $x = 2n\pi \pm \dfrac{\pi}{3}$, $n \in \mathbb{Z}$. Hence the domain is $\mathbb{R} - \left\{2n\pi \pm \dfrac{\pi}{3}\right\}, n \in \mathbb{Z}$.

3. Draw the graph of the functions $f(x) = |x|$, $f(x) = |x - 1|$ and $f(x) = |x + 1|$ **[Qy. - 2018]**

Sol :



$f(x) = |x - 1|$ causes the graph of the function $f(x) = |x|$ shifts to the right for one unit.

$f(x) = |x + 1|$ causes the graph of the function $f(x) = |x|$ shifts to the left for one unit.

4. If $f : R - (-1, 1) \to R$ is defined by $f(x) = \dfrac{x}{x^2 - 1}$, verify whether f is one to one. **[Qy. - 2018]**

Sol : We start with the assumption $f(x) = f(y)$.

Then $\dfrac{x}{x^2 - 1} = \dfrac{y}{y^2 - 1}$

$\Rightarrow \quad x(y^2 - 1) = y(x^2 - 1)$

$\Rightarrow \quad xy^2 - x - yx^2 + y = 0$

$\Rightarrow \quad (y - x)(xy + 1) = 0$

This implies that $x = y$ or $xy = -1$. So if we select two numbers x and y so that $xy = -1$, then $f(x) = f(y)$. $\left(2, -\frac{1}{2}\right)$ $\left(7, -\frac{1}{7}\right)\left(-2, \frac{1}{2}\right)$ are some among the infinitely many possible pairs. Thus $f(2) = f\left(\frac{-1}{2}\right) = \frac{2}{3}$. That is, $f(x) = f(y)$ does not imply $x = y$. Hence it is not one - to - one.

5. If A and B are two sets so that $[n(B - A) = 2\,n(A-B) = 4\,n(A\cap B)]$ and if $n(A\cup B) = 14$, then find $n[P(A)]$. **[First Mid - 2018]**

Sol : To find $n(P(A))$, we need $n(A)$

$$\begin{aligned} \text{Let } n(A\cap B) &= K. \\ \text{Then } n(A-B) &= 2k \text{ and } n(B-A) = 4k. \\ \text{Now } n(A\cup B) = n(A-B) &+ n(B-A) + n(A\cap B) = 7k \\ \text{It is given that } n(A\cup B) &= 14. \\ \text{Thus } 7k &= 14 \text{ and hence } k = 2. \\ \text{So, } n(A-B) &= 4 \text{ and } n(B-A) = 8. \\ \text{As } n(A) &= n(A-B) + n(A\cap B), \\ \text{we get } n(A) &= 6 \text{ and } n(P(A)) = 2^6 = 64. \end{aligned}$$

6. If the function f and g are given by $f = \{(1, 2), (3, 5), (4, 1)\}$ $g = \{(2, 3), (5, 1), (1, 3)\}$ find range of f and g. Also write down fog. **[First Mid - 2018]**

Sol : To check whether compositions can be defined, let us find the domain and range of these functions

$$\begin{aligned} \text{Domain of } f &= \{1, 3, 4\} \\ \text{Range of } f &= \{2, 5\} \\ \text{Domain of } g &= \{1, 2, 5\} \\ \text{Range of } g &= \{1, 3\} \end{aligned}$$

Since the range of f is contained in the domain of g we can define gof ; so as to find the image of 1 under gof, we first find the image of 1 under f and then its image under g. The image of 1 under f is 3 and its image under g is 1.

So $(gof)(1) = g(f(1)) = g(2) = 1$.

7. If $n(A) = 10$ and $n(A\cap B) = 3$ find $n[(A\cap B)' \cap A]$.

Sol : Refer Text Book Example 1.7 **[Aug. - 2022]**

8. Consider the functions **[Qy. - 2023]**

(i) $f(x) = x^2$ (ii) $f(x) = x^2 + 1$ (iii) $f(x) = (x+1)$

Sol : Refer Text Book Example 1.11

9. In the set Z of integers, define mRn if $m - n$ is a multiple of 12. Prove that R is an equivalence relation. **[July-'24]**

Sol : Refer Text Book Example 1.13

SECTION - D (5 MARKS)

1. A relation R is defined on the set z of integers as follows :
$(x, y) \in \mathbb{R} \Leftrightarrow x^2 + y^2 = 25$. Express R and R^{-1} as the set of ordered pairs and hence find their respective domains. **[Qy. - 2018]**

Sol :

$$\begin{aligned} x^2 + y^2 &= 25 \\ y &= \pm\sqrt{25 - x^2} \\ x &= 0 \Rightarrow y = \pm 5 \\ y &= 0 \Rightarrow x = \pm 5 \\ (0,5), (0,-5) &\in \mathbb{R} \\ x &= 3 \quad \Rightarrow y = \pm 4 \\ x &= -3 \quad \Rightarrow y = \pm 4 \\ \text{Domain of } \mathbb{R} &= \{0, 3, -3, -4, 4, -5, 5\} \\ \text{Domain of } \mathbb{R}^{-1} &= \{0, 3, -3, -4, 4, -5, 5\} \end{aligned}$$

2. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x - 3$, then prove that f is a bijection and find its inverse. **[Hy. - 2018; Qy. - 2019; Mar. - 2024]**

Sol : One-to-one : Let $f(x) = f(y)$. Then $2x - 3 = 2y - 3$; this implies that $x = y$. That is, $f(x) = f(y)$. implies that $x = y$: Thus f is one-to-one.

Onto : Let $y \in \mathbb{R}$. Let $x = \frac{y+3}{2}$.

Then $f(x) = \left(\frac{y+3}{2}\right) - 3 = y$. Thus f is onto. This also can be proved by saying the following statement. The range of f is R which is equal to the co-domain and hence f is onto.

Inverse Let $y = 2x - 3$. Then $y + 3 = 2x$ and hence $x = \frac{y+3}{2}$ Thus $f^{-1}(y) = \frac{y+3}{2}$ By replacing y as x, we get $f^{-1}(x) = \frac{x+3}{2}$.

3. If the function f is defined as

$$f(x) = \begin{cases} 3x - 2, & x > 3 \\ x^2 - 2, & -2 \le x \le 2 \\ 2x + 1, & x < -2 \end{cases}$$

Then find the values, if exists $f(4)$, $f(-4)$, $f(0)$, $f(-7)$. **[Hy. - 2018]**

Sol :

$$\begin{aligned} f(4) &= 3(4) - 2 = 10 \\ f(-4) &= 2(-4) + 1 = -7 \\ f(0) &= 0^2 - 2 = -2 \\ f(-7) &= 2(-7) + 1 = -13 \end{aligned}$$

4. Let A = {0, 1, 2, 3}. Construct relation on A of the following type. **[First Mid - 2018]**

(i) not reflexive, not symmetric, not transitive
(ii) not reflexive, not symmetric, transitive
(iii) not reflexive, symmetric, not transitive
(iv) not reflexive, symmetric, transitive
(v) reflexive, not symmetric, not transitive

Sol : (i) Let us use the pair (1, 2) to make the relation "not symmetric" and consider the relation {(1, 2)}. It is transitive. If we include (2, 3) and not include (1, 3), then the relation is not transitive. So the relation {(1, 2), (2, 3)} is not reflexive, not symmetric and not transitive.

(ii) Just now we have seen that the relation {(1, 2)} is transitive, not reflexive and not symmetric.

(iii) Let us start with the pair (1, 2). Since we need symmetricity, we have to include the pair (2, 1). At this stage as (1, 1), (2, 2) are not here, the relation is not transitive. Thus {(1, 2), (2, 1)} is not reflexive, it is symmetric; and it is not transitive.

(iv) If we include the pairs (1, 1) and (2, 2)to the relation discussed in (iii), it will become transitive. Thus {(1, 2), (2, 1), (1, 1), (2, 2)} is not reflexive; it is symmetric and it is transitive.

(v) For a relation on {0,1,2,3} to be reflexive, it must have the pairs (0, 0), (1, 1), (2, 2), (3, 3). Fortunately it becomes symmetric and transitive. Therefore, as in (i) if we insert (1, 2) and (2, 3) we get the required one.

Thus {(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is reflexive; it is not symmetric and it is not transitive.

5. In a survey of 5000 persons in a town, it was found that 45% of the person know language A, 25% know language B, 10% know language C, 5% know languages A and B, 4% know languages A and C. If 3% of the person know all the three languages find the number of persons who knows only language A. **[First Mid - 2018; May - 2022]**

Sol : This problem can be solved either by property of cardinality or by venn diagram.

Cardinality:

Given that $n(A) = 45\%$ of $5000 = 2250$

Similarly, $n(B) = 1250$, $n(C) = 500$.

$n(A \cap B) = 250$, $n(B \cap C) = 200$,

$n(C \cap A) = 200$ and $n(A \cap B \cap C) = 150$.

The number of persons who knows only language A is

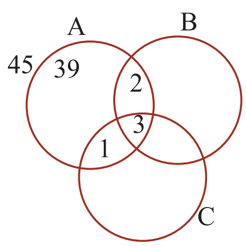
$$n(A \cap B' \cap C') = n\{A \cap (B \cup C)'\}$$
$$= n(A) - n\{A \cap (B \cup C)\}$$
$$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 2250 - 250 - 200 + 150 = 1950$$

Thus the required number of persons is 1950.

Venn diagram:

We draw the venn diagram using percentage.

The percentage of persons who knows only language A is 39. Therefore, the required number of persons is



$$\overset{50}{\cancel{5000}} \times \frac{39}{\cancel{100}} = 1950.$$

6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x - |x|$ and $g(x) = 2x + |x|$. Find fog. **[Govt. MQP - 2018; June & Hy. - 2019; Sep.- 2020; QY-'24**

Sol :

We know $|x| = \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$

So $f(x) = \begin{cases} 2x-(-x) & \text{if } x \le 0 \\ 2x - x & \text{if } x > 0 \end{cases}$

Thus $f(x) = \begin{cases} 3x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$

Also $g(x) = \begin{cases} 2x+(-x) & \text{if } x \le 0 \\ 2x + x & \text{if } x > 0 \end{cases}$

Thus $g(x) = \begin{cases} x; & x \le 0 \\ 3x; & x > 0 \end{cases}$

Let $x \le 0$.

Then $(gof)(x) = g(f(x)) = g(3x) = 3x$

The last equality is taken because $3x \le 0$ whenever $x \le 0$.

Let $x > 0$.

Then $(gof)(x) = g(f(x)) = g(x) = 3x$

Thus $(gof)(x) = 3x$ for all x.

7. Let A = {2, 3, 5} and relation R = {(2, 5)} write down the minimum number of ordered pairs to be included to R to make it an equivalence relation. **[Govt. MQP - 2018]**

Sol : It is enough to add (2, 2), (3, 3) and (5, 5) to make R reflexive.

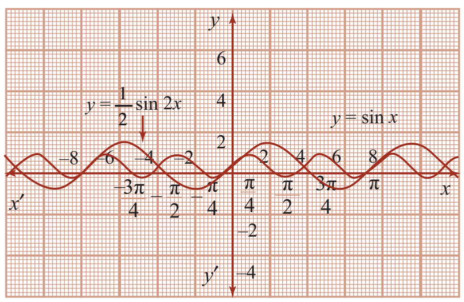
To make R symmetric, (5, 2) needs to be included.

Given R is a transitive relation.

∴ Minimum number of ordered pairs required are (5, 2), (2,2), (3,3) and (5, 5).

8. For the given base curve $y = \sin x$, draw $y = \frac{1}{2} \sin 2x$. **[Mar. - 2019]**

Sol : $y = \frac{1}{2}\sin 2x$



9. If $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = 2x^2 - 1$, find the pre-images of 17, 4 and –2. **[Hy. - 2019]**

Sol : To find the pre-image of 17, we solve the equation $2x^2 - 1 = 17$. The two solutions of this equation, 3 and −3 are the pre images of 17 under f. The equation $2x^2 - 1 = 4$ yields $\sqrt{\frac{5}{2}}$ and $-\sqrt{\frac{5}{2}}$ as the two pre-images of 4. To find the pre-image of −2, we solve the equation $2x^2 - 1 = -2$. This shows that $x^2 = -\frac{1}{2}$ which has no solution in $\mathbb{R}$ because square of a number cannot be negative and hence −2 has no pre-image under f.

10. If $y = \sin^{-1}\frac{1}{2}\left(\sqrt{1+x}+\sqrt{1-x}\right)$ then show that $\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$. **[April - 2023]**

Sol :

$$y = \sin^{-1}\frac{\sqrt{1+x}+\sqrt{1-x}}{2}$$

Put $x = \cos\theta$

$$\therefore y = \sin^{-1}\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}$$

$$= \sin^{-1}\frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}+\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{2}$$

$$= \sin^{-1}\frac{\sqrt{2}\left(\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right)}{2}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}+\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right)$$

$$= \sin^{-1}\left(\sin\frac{\pi}{4}\cos\frac{\theta}{2}+\cos\frac{\pi}{4}\sin\frac{\theta}{2}\right)$$

$$= \sin^{-1}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{\pi}{4}+\frac{\theta}{2}$$

$$\therefore y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{1-1}{2\sqrt{1-x^2}} = \frac{-1}{2\sqrt{1-x^2}}$$ Hence proved.

ADDITIONAL PROBLEMS

SECTION - A (1 MARK)

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER.

1. Which one of the following is a finite set?

(1) $\{x: x \in \mathbb{Z}, x < 5\}$ (2) $\{x: x \in \mathbb{W}, x \geq 5\}$
(3) $\{x: x \in \mathbb{N}, x > 10\}$
(4) $\{x: x$ is an even prime number$\}$

[Ans : (4) $\{x: x$ is an even prime number$\}$]

Hint : $\{x : x$ is an even prime number$\} = \{2\}$

2. If A ⊆ B, then A\B is

(1) B (2) A (3) ∅ (4) $\frac{B}{A}$

Hint : If A ⊆ B, then every element of A is element of B, So $\frac{A}{B}$ is ∅. **[Ans : (3) ∅]**

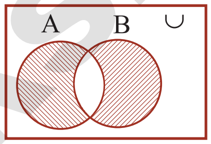
3. Given A = {5, 6, 7, 8}. Which one of the following is incorrect?

(1) ∅ ⊆ A (2) A ⊆ A
(3) {7, 8, 9} ⊆ A (4) {5} ⊂ A

Hint : 9 ∉ A, So {7, 8, 9} ⊈ A **[Ans : (3) {7, 8, 9} ⊆ A]**

4. The shaded region in the adjoining diagram represents.

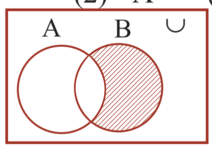
(1) A\B (2) B\A (3) A Δ B (4) A′



Hint : (A – B) ∪ (B – A) = A Δ B **[Ans : (3) A Δ B]**

5. The shaded region in the adjoining diagram represents.

(1) A\B (2) A′ (3) B′ (4) B\A



[Ans : (4) B\A]

6. Let R be a relation on the set $\mathbb{N}$ given by $\mathbb{R} = \{(a, b): a = b - 2, b > 6\}$. Then

(1) $(2, 4) \in \mathbb{R}$ (2) $(3, 8) \in \mathbb{R}$
(3) $(6, 8) \in \mathbb{R}$ (4) $(8, 7) \in \mathbb{R}$

Hint : $6 = 8 - 2 \Rightarrow 6 = 6$ **[Ans : (3) $(6, 8) \in \mathbb{R}$]**

7. For real numbers x and y, define xRy if $x - y + \sqrt{2}$ is an irrational number. Then the relation R is

(1) reflexive (2) symmetric
(3) transitive (4) none of these

Hint : x R $x \Rightarrow \cancel{x} - \cancel{x} + \sqrt{2} = \sqrt{2}$, irrational R is reflexive. **[Ans : (1) reflexive]**

8. Let R be the relation over the set of all straight lines in a plane such that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$. Then R is

(1) symmetric (2) reflexive
(3) transitive (4) an equivalence relation

Hint : $l_1 \perp l_2 \Rightarrow l_2 \perp l_1$ **[Ans : (1) symmetric]**
$\Rightarrow$ R is symmetric

9. Which of the following is not an equivalence relation on z?

(1) aR$b \Leftrightarrow a + b$ is an even integer
(2) aR$b \Leftrightarrow a - b$ is an even integer
(3) aR$b \Leftrightarrow a < b$ (4) aR$b \Leftrightarrow a = b$

Hint : a is not less than b. **[Ans : (3) aR$b \Leftrightarrow a < b$]**
$\therefore a$R$b \Leftrightarrow a < b$ is not an equivalence relation.

10. Which of the following functions from z to itself are bijections (one-one and onto)?

(1) $f(x) = x^3$ (2) $f(x) = x + 2$

(3) $f(x) = 2x + 1$ (4) $f(x) = x^2 + x$

[Ans : (2) $f(x) = x + 2$]

11. Let f: Z → Z be given by $f(x) = \begin{cases} \frac{x}{2} \text{ if } x \text{ is even} \\ 0 \text{ if } x \text{ is odd} \end{cases}$. Then f is

(1) one-one but not onto

(2) onto but not one-one

(3) one-one and onto

(4) neither one-one nor onto

[Ans : (2) onto but not one-one]

Hint : $f(3) = f(5) = 0$. Hence f is not one-one.

12. If f: R → R is given by $f(x) = 3x - 5$, then $f^{-1}(x)$ is

(1) $\frac{1}{3x-5}$ (2) $\frac{x+5}{3}$

(3) does not exist since f is not one-one

(4) does not exists since f is not onto

[Ans : (2) $\frac{x+5}{3}$]

Hint :

$$y = 3x - 5 \Rightarrow \frac{y+5}{3} = x$$

$$\Rightarrow g(y) = \frac{y+5}{3} \Rightarrow g(x) = \frac{x+5}{3}$$

13. If $f(x) = 2x - 3$ and $g(x) = x^2 + x - 2$ then $gof(x)$ is

(1) $2(2x^2 - 5x + 2)$ (2) $(2x^2 - 5x - 2)$

(3) $2(2x^2 + 5x + 2)$ (4) $2x^2 + 5x - 2$

Hint :

$$gof(x) = (2x-3)^2 + 2x - 3 - 2$$
$$= 4x^2 + 9 - 12x + 2x - 3 - 2$$
$$= 1\,(2x^2 - 5x + 2)$$

[Ans : (1) $2(2x^2 - 5x + 2)$]

14. Let f: ℝ → ℝ be given by $f(x) = x + \sqrt{x^2}$ is

(1) injective (2) subjective

(3) bijective (4) none of these

[Ans : (4) none of these]

Hint : f is neither one - one nor onto.

15. Choose the correct statement.

(1) One-to-one function have inverse

(2) Onto function have inverse

(3) bijection function have inverse

(4) many - to - one function have inverse

[Ans : (3) bijection function have inverse]

16. Match List - I with List II

	List I		List II
i.	{(1, 1), (2,2), (3,3)(1,2)}	(a)	equivalence
ii.	{(1,2), (2,1), (2,3), (3,2)}	(b)	transitive
iii.	{(1,2), (2,3), (1,3)}	(c)	Symmetric
iv.	{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (1,3)}	(d)	reflexive

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	c	d	b	a
(2)	d	c	b	a
(3)	b	a	d	c
(4)	b	a	b	c

[Ans : (2) i – d ii – c iii – b iv – a]

SECTION - C (3 MARKS)

1. Find the pairs of equal sets from the following sets. A = {0}, B = {x: x > 15 and x < 5}, C = {x: x – 5 = 0}, D = {x: x^2 = 25}, E = {x: x is an integral positive root of the equation $x^2 - 2x - 15 = 0$}.

Sol :

Given A = {0} ...(1)

B = {x: x > 15 and x < 5}

⇒ B = φ ...(2)

C = {x: x – 5 = 0}

⇒ C = {5} ...(3)

D = {x: x^2 = 25}

⇒ D = {–5, 5} ...(4)

E = {x: x is an integral positive root of $x^2 - 2x - 15 = 0$}

⇒ E = {5} ...(5)

From (3) and (5), clearly C = E.

Hence C and E are equal sets.

2. Construct a suitable domain X such that f : X → N defined by $f(n) = n + 3$ to be one to one and onto.

Sol :

f : X → N

$f(n) = n + 3$

∴ Suitable domain :

X = {–2, –1, 0} ∪ N

i.e., X = {–2, –1, 0, 1, 2,.....}

02 BASIC ALGEBRA

MUST KNOW DEFINITIONS

Rational numbers :	Any number of the form $\frac{p}{q}$, where $q \neq 0$ is called a rational number where $p, q \in z$.
Irrational numbers :	A number that cannot be expressed as a ratio between two integers and is not an imaginary number.
Intervals :	✦ If a, b are real numbers such that $a < b$, then the set $\{x: a < x < b\}$ is called the open interval from a to b i.e. (a, b). ✦ The set $\{x: a \leq x \leq b\}$ is called the closed interval from a to b and is written as $[a, b]$ ✦ If a is any real number, then the sets of the type $\{x: x < a\}$, $\{x: x \leq a\}$, $\{x: x > a\}$ and $\{x: x \geq a\}$ are called infinite intervals and are respectively written as $(-\infty, a)$, $(-\infty, a]$, (a, ∞) and $[a, \infty)$. These are semi-open and semi-closed intervals.
Absolute value of x :	Absolute value of $x = \lvert x \rvert$ is defined as: $\lvert x \rvert = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
Radical (Surd) :	If 'a' is a positive rational number and n is a positive integer such that $\sqrt[n]{a}$ is an irrational number then $\sqrt[n]{a}$ is called a surd or a radical.
Mixed surd :	A surd is called a mixed if its rational co-efficient is other than unity. If the product of two irrational numbers is rational, then each one is called the rationalizing.
Pure surd :	A surd is a pure surd if its rational co-efficient is unity.
Polynomial :	An expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable
Identity :	An identity is a statement of equality between two expressions which is true for all values of the variable involved.
Equation :	An equation is a statement of equality between two expressions which is not true for all values of the variable involved.

Root of an equation : A value of the variable for which an equation is satisfied is called a root of an equation.

Logarithm : If $a > 0$, $a \neq 1$ and $a^x = y$, then we define the logarithm of y to the base a as x and written as $\log_a y = x$.

Common logarithm : The logarithm to the base '10' are called **Common logarithm.**

Disjoint set : Two sets A and B are said to be disjoint if there is no element common to both A and B.

Characteristic : The integral part of the common logarithm of a number., is characteristic
It may be either positive or zero or negative.

Mantissa : The positive decimal part of the common logarithm of a number is Mantissa
It may be either positive or zero.

Partial fractions :

(i) For Linear factors: Rational expression of the form "$\frac{p}{q}$ where q is the non repeated product of linear factors like $(ax + b)(cx + d)$ can be written as $\frac{M}{ax+b} + \frac{N}{cx+d}$.

(ii) $\frac{p}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$

(iii) $\frac{p}{ax^2+bx+c} = \frac{Ax+B}{ax^2+bx+c}$ where p is a rational expression of degree less than the denominator

Formulae to Remember :

Laws of Radicals : For positive integers m, n and positive rational numbers a, b we have

(i) $\left(\sqrt[n]{a}\right)^n = a = \sqrt[n]{a^n}$ (ii) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

(iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ (iv) $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

Inequality : If $a > b$, then

(i) $a + c > b + c$ for any $c \in$ R

(ii) $a - c > b - c$ for any $c \in$ R

(iii) $-a < -b$

(iv) $ac > bc$, $\frac{a}{c} > \frac{b}{c}$ for any positive real number c.

(v) $ac < bc$, $\frac{a}{c} < \frac{b}{c}$ for any negative real number c.

(vi) If $a < b$ then $\frac{1}{a}. > \frac{1}{b}$

Identities :
(i) $(x + a)(x + b) = x^2 + x(a + b) + ab$
(ii) $(a + b)^2 = a^2 + 2ab + b^2$
(iii) $(a - b)^2 = a^2 - 2ab + b^2$
(iv) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(v) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
(vi) $(a + b)(a - b) = a^2 - b^2$
(vii) $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ (OR) $(a + b)(a^2 - ab + b^2)$
(viii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b)$ (OR) $(a - b)(a^2 + ab + b^2)$
(ix) $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ldots + 1)$

Quadratic equation : The roots of $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Laws of logarithm :
(i) **Product rule:** $\log_a mn = \log_a m + \log_a n$
(ii) **Quotient rule:** $\log_a (m/n) = \log_a m - \log_a n$
(iii) **Power rule:** $\log_a nm = n \cdot \log_a m$
(iv) **Change of base rule:** $\log_a m = \log_b m \times \log_a b$
Also $\log_a b \times \log_a b = 1$

Basic Results :
(v) If $a > 0$, $a \neq 1$, and $a^x = y$, then $\log_a y = x$
(vi) $\log_a x_1 = \log_a x_2 \Rightarrow x_1 = x_2$
(vii) $\log_a 1 = 0$ and $\log_a a = 1$

Absolute value : If a is a positive real number, then
(i) $|x| < a \Leftrightarrow x \in (-a, a)$ (ii) $|x| \leq a \Leftrightarrow x \in [-a, a]$
(iii) $|x| > a \Leftrightarrow x < -a$ or $x > a$ (iv) $|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$

TEXTUAL QUESTIONS

Exercise 2.1

1. Classify each element of $\left\{\sqrt{7}, \dfrac{-1}{4}, 0, 3.14, 4, \dfrac{22}{7}\right\}$ as a member of $\mathbb{N}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}$ or $\mathbb{Z}$.

Sol : Since $\sqrt{7}$ is an irrational number, $\sqrt{7} \in \mathbb{R} - \mathbb{Q}$.

Since $\dfrac{-1}{4}$ is a negative rational number, $\dfrac{-1}{4} \in \mathbb{Q}$

0 is an integer and $0 \in \mathbb{Z}, \mathbb{Q}$

$3.14 = \pi$ is a non-recurring and non terminating decimal.

$\therefore$ 3.14 is an irrational number $\Rightarrow 3.14 \in \mathbb{Q}$

4 is a positive integer $\Rightarrow 4 \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}$.

$\dfrac{22}{7} \in \mathbb{Q}$ Which is an irrational number.

2. Prove that $\sqrt{3}$ is an irrational number. (Hint: Follow the method that we have used to prove $\sqrt{2} \notin \mathbb{Q}$.) **[First Mid - 2018]**

Sol : Suppose $\sqrt{3}$ is a rational number.

Then $\sqrt{3}$ can be written as $\sqrt{3} = \dfrac{m}{n}$

Where m and n are positive integers with no common factors other than 1.

Squaring both sides we get,

$$3 = \frac{m^2}{n^2} \Rightarrow 3n^2 = m^2$$

Multiplying by 2 we get,

$$6n^2 = 2m^2 \Rightarrow 3(2n^2) = 2m^2 \quad \ldots(1)$$

Since $2n^2$ is divisible by 2, m^2 is also an even number

$\Rightarrow m$ must be even

$\Rightarrow \quad m = 2k$ for some natural number k

$\Rightarrow \quad 3n^2 = (2k)^2$

$\Rightarrow \quad 3n^2 = 4k^2$ [From (1)]

$\Rightarrow n$ is also an even number.

Thus both m and n are even numbers having a common factor 2.

This contradicts our initial assumption that m and n do not have a common factor.

Hence $\sqrt{3}$ cannot be a rational number.

$\Rightarrow \sqrt{3}$ is an irrational number.

Hence proved.

3. Are there two distinct irrational numbers such that their difference is a rational number? Justify.

Sol : Let the two distinct irrational numbers be $(2+\sqrt{3})$ and $(4+\sqrt{3})$.

Their difference is $(2+\sqrt{3})-(4+\sqrt{3})$

$= 2 + \sqrt{3} - 4 - \sqrt{3} = 2 - 4 = -2$ which is rational.

4. Find two irrational numbers such that their sum is a rational number. Can you find two irrational numbers whose product is a rational number.

Sol : Let the two irrational numbers be

$5+\sqrt{7}$ and $7-\sqrt{7}$.

Their sum $= (5+\sqrt{7})+(7-\sqrt{7}) = 5+\sqrt{7}+7-\sqrt{7}$

$= 5 + 7 = 12$ which is a rational number.

Consider the two irrational numbers

$4+\sqrt{6}$ and $4-\sqrt{6}$.

Their product $= (4+\sqrt{6})+(4-\sqrt{6}) = 4^2-(\sqrt{6})^2$

$= 16 - 6 = 10$ which is a rational number.

5. Find a positive number smaller than $\frac{1}{2^{1000}}$. Justify.

Sol : Given number is $\frac{1}{2^{1000}}$.

We know $1000 < 1001$

$\Rightarrow 2^{1000} < 2^{1001} \Rightarrow \frac{1}{2^{1000}} > \frac{1}{2^{1001}}$

$\therefore$ A positive number smaller than $\frac{1}{2^{1000}}$ is $\frac{1}{2^{1001}}$.

Exercise 2.2

1. Solve for x

(i) $|3-x| < 7$ **(ii)** $|4x-5| \geq -2$

(iii) $\left|3-\frac{3}{4}x\right| \leq \frac{1}{4}$ **(iv)** $|x| - 10 < -3$

Sol : (i) $|3-x| < 7$

Given $|3-x| < 7$

This means $-7 < 3-x < 7$

$\Rightarrow -7-3 < -x < 7-3 \qquad \Rightarrow -10 < -x < 4$

$\Rightarrow 10 > x > -4$ [$a < b \Rightarrow ay > by$ for all $y < 0$]

$\Rightarrow -4 < x < 10$ Here $y = -1$.

(ii) $|4x-5| \geq -2$

$\Rightarrow 4\left|x-\frac{5}{4}\right| \geq -2 \Rightarrow \left|x-\frac{5}{4}\right| \geq -\frac{2}{4}$

Any $x \in$ R will satisfy this inequality.

(iii) $\left|3-\frac{3}{4}x\right| \leq \frac{1}{4}$

This means $-\frac{1}{4} \leq 3-\frac{3}{4}x \leq \frac{1}{4}$.

$\Rightarrow -\frac{1}{4}-3 \leq -\frac{3}{4}x \leq \frac{1}{4}-3$

$\Rightarrow -\frac{13}{4} \leq -\frac{3}{4}x \leq -\frac{11}{4}$

$\Rightarrow 13 \geq 3x \geq 11 \Rightarrow \frac{11}{3} \leq x \leq \frac{13}{3}$

(iv) $|x| - 10 < -3$

Given $|x| - 10 < -3$

$\Rightarrow |x| < -3 + 10$

$\Rightarrow |x| < 7$

This means $-7 < x < 7$.

2. Solve $\frac{1}{|2x-1|} < 6$ and express the solution using the interval notation.

Sol : Given $\frac{1}{|2x-1|} < 6$

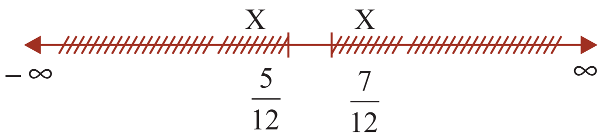
Multiplying the numerator and denominator by $|2x-1|$ we get, $\frac{|2x-1|}{|2x-1|^2} < 6$

$\Rightarrow \quad 1 < 6\,|2x-1| \qquad \Rightarrow \quad 0 < 6\,|2x-1| - 1$

$\Rightarrow \quad 6\,|2x-1| - 1 > 0$

$\Rightarrow \quad \pm 6\,(2x-1) - 1 > 0$

$6(2x-1)-1 > 0$	$-6(2x-1)-1 > 0$
$12x-6-1 > 0$	$-12x+6-1 > 0$
$12x-7 > 0$	$-12x+5 > 0$
$12x > 7$	$-12x > -5$
$x > \frac{7}{12}$	$x < \frac{5}{12}$



$\therefore$ The solution set is $\left(-\infty, \frac{5}{12}\right) \cup \left(\frac{7}{12}, \infty\right)$

3. Solve –3 |*x*| + 5 ≤ –2 and graph the solution set in a number line.

Sol : Given $-3\,|x| + 5 \le -2 \Rightarrow -3\,|x| \le -2 - 5 \Rightarrow -3\,|x| \le -7$

$\Rightarrow |x| \ge \frac{7}{3}$ [Dividing by –3]

This means $-\frac{7}{3} \ge |x| \ge \frac{7}{3}$

$\therefore$ The solution set is $\left(-\infty, \frac{-7}{3}\right] \cup \left[\frac{7}{3}, \infty\right)$.

4. Solve 2|*x* + 1| – 6 ≤ 7 and graph the solution set in a number line. **[Hy. - 2018]**

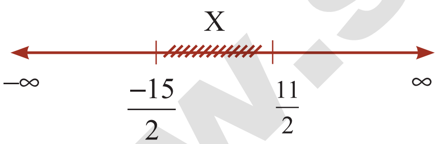
Sol : Given $2\,|x + 1| - 6 \le 7$

$\Rightarrow \; 2\,|x + 1| \le 7 + 6 \Rightarrow 2\,|x + 1| \le 13$

$\Rightarrow \; |x + 1| \le \frac{13}{2}$.

This means $\frac{-13}{2} \le x + 1 \le \frac{13}{2}$

$\Rightarrow \; \frac{-13}{2} - 1 \le x \le \frac{13}{2} - 1 \Rightarrow \frac{-15}{2} \le x \le \frac{11}{2}$.



$\therefore$ The solution set is $\left[\frac{-15}{2}, \frac{11}{2}\right]$.

5. Solve: $\frac{1}{5}|10x - 2| < 1$.

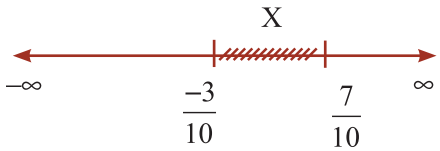
Sol : Given inequality is $\frac{1}{5}|10x - 2| < 1$.

$\Rightarrow \; |10x - 2| < 5$

This means, $-5 < 10x - 2 < 5$.

$\Rightarrow \; -5 + 2 < 10x < 5 + 2 \quad \Rightarrow -3 < 10x < 7$

$\Rightarrow \; -\frac{3}{10} < x < \frac{7}{10}$.



$\therefore$ Solution set is $\left(\frac{-3}{10}, \frac{7}{10}\right)$.

6. Solve: |5*x* – 12| < –2.

Sol : $-(-2) < 5x - 12 < -2$.

$+2 + 12 < 5x < -2 + 12$

$14 < 5x < 10$

$\frac{14}{5} < x < 2$

$2.8 < x < 2$, which is not possible.

Hence no solution.

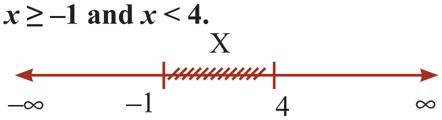
Exercise 2.3

1. Represent the following inequalities in the interval notation:

(i) *x* ≥ –1 and *x* < 4 (ii) *x* ≤ 5 and *x* ≥ –3

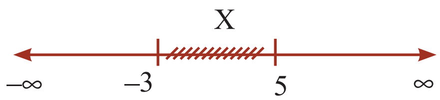
(iii) *x* < –1 or *x* < 3 (iv) –2*x* > 0 or 3*x* – 4 < 11

Sol : (i) **$x \ge -1$ and $x < 4$.**



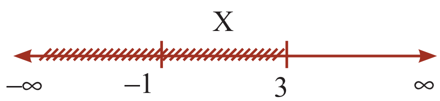
$\Rightarrow x \in [-1, 4)$

(ii) **$x \le 5$ and $x \ge -3$**



$\Rightarrow x \in [-3, 5]$

(iii) **$x < -1$ or $x < 3$**



$\Rightarrow x \in (-\infty, 3)$

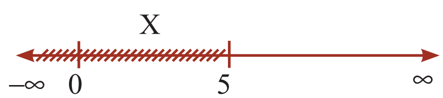
(iv) **$-2x > 0$ or $3x - 4 < 11$**

$\Rightarrow \; -x > 0$ or $3x < 11 + 4$

$\Rightarrow \; x < 0$ or $3x < 15$ $[a > b \Rightarrow -a < -b]$

$\Rightarrow \; x < 0$ or $x < \frac{15}{3} \Rightarrow x < 0$ or $x < 5$

$\Rightarrow \; x \in (-\infty, 5)$



2. Solve $23x < 100$ when [July-2023 & '24; QY-'24]
(i) x is a natural number (ii) x is an integer.

Sol : Given $23x < 100$.

(i) when x is a natural number $23x < 100$

$\Rightarrow x < \frac{100}{23} \Rightarrow x < 4.348 \Rightarrow x = \{1, 2, 3, 4\}$

(ii) when x is an integer $x < 4.348$

$\Rightarrow x = \{\cdots\cdots -3, -2, -1, 0\ 1, 2, 3, 4\}$

Hence solution set is $\{\cdots\cdots -3, -2, -1, 0\ 1, 2, 3, 4\}$.

3. Solve $-2x \geq 9$ when
(i) x is a real number (ii) x is an integer
(iii) x is a natural number. [CRT - 2022]

Sol : Given $-2x \geq 9 \Rightarrow -x \geq \frac{9}{2} \quad \Rightarrow x \leq \frac{-9}{2}$

(i) when x is a real number $x \in \left(-\infty, \frac{-9}{2}\right]$

(ii) when x is an integer $x \in \{\cdots, -7, -6, -5\}$

(iii) x is natural number

$x = \{\ \}$. Since there is no solution.

4. Solve : (i) $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$ [Qy. - 2023]

(ii) $\frac{5-x}{3} < \frac{x}{2} - 4$

Sol : (i) $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$

Given : $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$

$\Rightarrow \frac{3x-6}{5} \leq \frac{10-5x}{3}$

$\Rightarrow 3(3x-6) \leq 5(10-5x)$

$\Rightarrow 9x - 18 \leq 50 - 25x$

$\Rightarrow 9x + 25x \leq 50 + 18 \quad \Rightarrow 34x \leq 68$

$\Rightarrow x \leq 2$

$\therefore$ Solution set is $(-\infty, 2]$.

(ii) $\frac{5-x}{3} < \frac{x}{2} - 4$

Multiplying by 3, throughout, $5 - x < \frac{3x}{2} - 12$

Multiplying by 2, we get,

$10 - 2x < 3x - 24.$

$\Rightarrow 10 + 24 < 3x + 2x$

$34 < 5x \Rightarrow 5x > 34 \Rightarrow x > \frac{34}{5}$

$\Rightarrow x > 6.8$

$\therefore$ Solution set is $(6.8, \infty)$

5. To secure A grade one must obtain an average of 90 marks or more in 5 subjects each of maximum 100 marks. If one scored 84, 87, 95, 91 in first four subjects, what is the minimum mark one scored in the fifth subject to get A grade in the course? [First Mid - 2018]

Sol : Let the person obtain x marks in the fifth examination.

Then $\frac{84+87+95+91+x}{5} \geq 90 \Rightarrow \frac{357+x}{5} \geq 90$

Multiplying both sides by 5 we get,

$357 + x \geq 450$

Subtracting 357 from both sides, we get,

$x \geq 450 - 357. \Rightarrow x \geq 93$

Thus, the person must obtain a minimum of 93 marks to get A grade in the Course.

6. A manufacturer has 600 litres of a 12 percent solution of acid. How many litres of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more than 15 percent but less than 18 percent?

Sol : Let x be the number of litres of 30% acid solution.

$\therefore$ Total mixture $= (600 + x)$ litres

30% of x + 12% of 600 > 15% of $(600 + x)$

$\Rightarrow \frac{30x}{100} + \frac{12}{100} \times 600 > \frac{15}{100}(600 + x)$

$\Rightarrow 30x + 7200 > 9000 + 15x$ [Multiplying by 100]

$\Rightarrow 30x + 7200 - 15x > 9000$ [Subtracting $15x$]

$\Rightarrow 15x + 7200 > 9000 \Rightarrow 15x > 9000 - 7200$

$\Rightarrow 15x > 1800 \quad \Rightarrow x > 120$ ...(1)

Also, 30% of x + 12% of 600 < 18% of $(600 + x)$

$\Rightarrow \frac{30x}{100} + \frac{12}{100} \times 600 < \frac{18}{100}(600 + x)$

$\Rightarrow 30x + 7200 < 18\,(600 + x)$ [Multiplying by 100]

$\Rightarrow 30x + 7200 < 10,800 + 18x$

$\Rightarrow 12x + 7200 < 10,800$ [Subtracting $18x$]

$\Rightarrow 12x < 10,800 - 7200$ [Subtracting 7200]

$\Rightarrow 12x < 3600$

$\Rightarrow x < \frac{3600}{12} \quad \Rightarrow x < 300$ ...(2)

From (1) and (2), $120 < x < 300$.

Thus, the number of litres of the 30% acid solution will have to be greater than 120 litres and less than 300 litres.

7. Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.

Sol : Let x be the smaller of two positive odd integers, so that other one is $x + 2$

Given $x > 10$, and $x + 2 > 10$. ...(1)

$\Rightarrow x > 10 - 2 \quad \Rightarrow x > 8$ ...(2)

And $(x) + (x + 2) < 40$ ...(3)

From (1) and (2) we get, $x > 10$ ...(4)

From (3) we get

$$2x + 2 < 40$$

$$\Rightarrow 2x < 40 - 2 \quad \Rightarrow 2x < 38$$

$$\Rightarrow x < \frac{38}{2} \quad \Rightarrow x < 19 \quad ...(5)$$

From (4) and (5) we get, $10 < x < 19$.

Since x is an odd natural number, x can take the values 11, 13, 15, 17.

Hence the required possible consecutive pairs will be (11, 13), (13, 15), (15, 17) (17, 19)

8. A model rocket is launched from the ground. The height 'h' reached by the rocket after t seconds from lift off is given by $h(t) = -5t^2 + 100t, 0 \le t \le 20$. At what time the rocket is 495 feet above the ground?

Sol : $h(t) = -5t^2 + 100t, 0 \le t \le 20$

Let the time be 't' sec, when the rocket is 495 feet above the ground

$$h(t) = -5t^2 + 100t = 495$$

$$\Rightarrow -5t^2 + 100t - 495 = 0$$

$$\Rightarrow t^2 - 20t + 99 = 0 \quad \text{[Divided by } -5]$$

$$\Rightarrow (t - 11)(t - 9) = 0 \quad \Rightarrow t = 11 \text{ or } 9.$$

$\therefore$ At 11 or 9 sec, the rocket is 495 feet above the ground.

9. A plumber can be paid according to the following schemes: In the first scheme he will be paid rupees 500 plus rupees 70 per hour, and in the second scheme he will paid rupees 120 per hour. If he works x hours, then for what value of x does the first scheme give better wages?

Sol : Let the number of hours to complete the job is x.

Wages from the first scheme = ₹(500 + 70x)

Wages from the second scheme = ₹120x

Given $500 + 70x > 120x$

$$\Rightarrow 500 > 120x - 70x$$

$$\Rightarrow 500 > 50x \quad \Rightarrow \frac{\overset{10}{\cancel{500}}}{\cancel{50}} > x$$

$$\Rightarrow 10 > x \quad \Rightarrow x < 10$$

$\therefore$ Number of hours should be less than ten hours.

10. A and B are working on similar jobs but their annual salaries differ by more than ₹ 6000. If B earns rupees 27000 per month, then what are the possibilities of A's salary per month?

Sol : Let A's salary be x. B's salary is ₹27,000

Given their difference in salary is more than ₹6000.

Assume A's salary is more than B's salary.

$$x - 27{,}000 > 6{,}000$$

$$\Rightarrow x > 6{,}000 + 27{,}000$$

$$\Rightarrow x > 33{,}000 \quad ...(1)$$

Assume B's salary is more than A's salary

$$\therefore ₹\ 27{,}000 - x > 6{,}000$$

$$\therefore ₹\ 27{,}000 - 6{,}000 > x$$

$$\Rightarrow x < 21{,}000 \quad ...(2)$$

From (1) and (2),

The possibilities of A's salary are greater than ₹ 33,000 or less than ₹ 21,000.

EXERCISE 2.4

1. Construct a quadratic equation with roots 7 and –3. **[CRT - 2022; Qy. - 2023]**

Sol : Given roots are 7 and –3

Sum of the roots $\alpha + \beta = 7 + (-3) = 4$

Product of the roots $\alpha\beta = 7(-3) = -21$.

The quadratic equation is $x^2 -$ (Sum of the roots) x + Product of the roots = 0

$$\Rightarrow x^2 - 4x - 21 = 0$$

Hence, the required quadratic equation is

$$x^2 - 4x - 21 = 0$$

2. A quadratic polynomial has one of its zeros as $1 + \sqrt{5}$ and it satisfies $p(1) = 2$. Find the quadratic polynomial.

Sol : Let the quadratic polynomial be

$$p(x) = k\,(x^2 - \text{(sum of the roots)}\,x + \text{Product of the roots})$$

Since $(1 + \sqrt{5})$ is a root, $(1 - \sqrt{5})$ is also a root

$$\therefore \text{Sum of the roots} = 1 + \sqrt{5} + 1 - \sqrt{5} = 2$$

$$\text{Product of the roots} = (1 + \sqrt{5})(1 - \sqrt{5}) = 1 - 5 = -4$$

$$\text{Polynomial is } p(x) = k\,(x^2 - 2x - 4)$$

$$\text{Also, given } p(1) = 2$$

$$k\,(1^2 - 2 - 4) = 2$$

$$k(-5) = 2 \Rightarrow k = -\frac{2}{5}$$

$$\therefore p(x) = -\frac{2}{5}(x^2 - 2x - 4)$$

3. If α and β are the roots of the quadratic equation $x^2+\sqrt{2}x+3=0$, form a quadratic polynomial with zeroes $\frac{1}{\alpha}$, $\frac{1}{\beta}$.

Sol : Given quadratic equation is $x^2+\sqrt{2}x+3=0$

$$\therefore \alpha+\beta=\frac{-b}{a}=-\sqrt{2} \Rightarrow \alpha\beta=\frac{3}{1}=3 \quad \text{... (1)}$$

$$\left[\because \; a=1,\quad b=\sqrt{2},\quad c=3 \quad \alpha+\beta=\frac{-b}{a},\quad \alpha\beta=\frac{c}{a}\right]$$

Sum of the roots

$$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta}=\frac{-\sqrt{2}}{3}$$

Product of the roots

$$\frac{1}{\alpha}\cdot\frac{1}{\beta} = \frac{1}{\alpha\beta}=\frac{1}{3}$$

Hence, the required quadratic equation is x^2 – (Sum of the roots) x + product of the roots = 0.

$$\Rightarrow x^2-\left(\frac{-\sqrt{2}}{3}\right)x+\frac{1}{3}=0 \quad \Rightarrow x^2+\frac{\sqrt{2}}{3}x+\frac{1}{3}=0$$

Multiplying by 3 we get, $3x^2+\sqrt{2}x+1=0$

4. If one root of $k(x-1)^2 = 5x-7$ is double the other root, show that $k = 2$ or -25.

[Govt. MQP - 2018; June - 2019; CRT - 2022; April - 2023]

Sol : Given equation is $k(x-1)^2 = 5x-7$

$$\Rightarrow k(x^2-2x+1) = 5x-7$$

$$\Rightarrow kx^2-2kx+k-5x+7 = 0$$

$$\Rightarrow kx^2+x(-2k-5)+(k+7) = 0$$

Let the roots be α and 2α.

$$\therefore \alpha+2\alpha = \frac{+2k+5}{k} \Rightarrow 3\alpha = \frac{+2k+5}{k}$$

$$\Rightarrow \alpha = \frac{+2k+5}{3k} \quad \text{... (1)}$$

$$\left[\alpha(2\alpha) = \frac{k+7}{k}\right]$$

$$\Rightarrow 2\alpha^2 = \frac{k+7}{k} \Rightarrow \alpha^2 = \frac{k+7}{2k} \quad \text{...(2)}$$

Substituting (1) in (2) we get,

$$\left(\frac{2k+5}{3k}\right)^2 = \frac{k+7}{2k} \Rightarrow \frac{4k^2+25+20k}{9k^2}=\frac{k+7}{2k}$$

$$\Rightarrow \frac{4k^2+25+20k}{9k}=\frac{k+7}{2}$$

$$\Rightarrow 8k^2+50+40k = 9k^2+63k$$

–50
23
25 –2

$$\Rightarrow k^2+23k-50=0$$

$$\Rightarrow (k-2)(k+25)=0 \Rightarrow k=2 \text{ or } -25.$$

Hence proved.

5. If the difference of the roots of the equation $2x^2-(a+1)x+a-1=0$ is equal to their product, then prove that $a = 2$. **[First Mid - 2018]**

Sol : Given equation is $2x^2-(a+1)x+a-1=0$.

Let α and β be the roots of the equation.

Given $\alpha-\beta = \alpha\beta$

$$\Rightarrow \alpha-\beta = \frac{a-1}{2} \quad \left[\because a=2,\ b=-(a+1),\ c=a-1.\ \text{Hence } \alpha\beta=\frac{c}{a}=\frac{a-1}{2}\right] \text{...(1)}$$

$$\text{and } \alpha+\beta = \frac{-b}{a}=\frac{a+1}{2} \quad \text{...(2)}$$

$$(1)+(2) \rightarrow \alpha-\beta = \frac{a-1}{2}$$

$$\alpha+\beta = \frac{a+1}{2}$$

$$2\alpha = \frac{a-1+a+1}{2}=\frac{2a}{2}=a$$

$$\boxed{\therefore \alpha = \frac{a}{2}}$$

Substituting $\alpha=\frac{a}{2}$ in (2) we get,

$$\frac{a}{2}+\beta = \frac{a+1}{2} \Rightarrow \beta=\frac{a+1}{2}-\frac{a}{2}=\frac{1}{2}$$

$$\boxed{\therefore \beta = \frac{1}{2}}$$

Substituting the values of α and β in $\alpha-\beta=\alpha\beta$

$$\text{we get } \frac{a}{2}-\frac{1}{2} = \left(\frac{a}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{a-1}{2} = \frac{a}{4} \Rightarrow a-1=\frac{a}{2}$$

$$\Rightarrow 2a-2 = a \Rightarrow 2a-a=2$$

$$\Rightarrow a = 2. \quad \text{Hence proved.}$$

6. Find the condition that one of the roots of ax^2+bx+c may be (i) negative of the other, (ii) thrice the other, (iii) reciprocal of the other.

Sol : (i) **negative of the other**

Given quadratic equation is $ax^2+bx+c=0$.

Since one root is negative of the other, let α and –α be the roots.

$$\therefore \alpha+(-\alpha) = \frac{-b}{a} \Rightarrow 0=\frac{-b}{a}$$

$\Rightarrow \quad b = 0$

Also $\alpha(-\alpha) = \frac{c}{a} \Rightarrow -\alpha^2 = \frac{c}{a}$

Hence the required condition is $b = 0$.

(ii) thrice the other

The roots are thrice the other.

Let α and 3α be the roots

$\alpha + 3\alpha = -\frac{b}{a} \quad \Rightarrow 4\alpha = -\frac{b}{a}$

$\Rightarrow \quad \alpha = -\frac{b}{4a}$ ... (1)

$(\alpha)(3\alpha) = \frac{c}{a} \quad \Rightarrow 3\alpha^2 = \frac{c}{a}$

$\Rightarrow \quad \alpha^2 = \frac{c}{3a}$ ...(2)

Substituting (1) in (2) we get,

$\left(\frac{-b}{4a}\right)^2 = \frac{c}{3a} \quad \Rightarrow \frac{b^2}{16a^2} = \frac{c}{3a}$

$\Rightarrow \quad \frac{b^2}{16a} = \frac{c}{3} \quad \Rightarrow \boxed{3b^2 = 16ac}$

Which is the required condition.

(iii) reciprocal of the other

The roots are reciprocal of the other.

Let α and $\frac{1}{\alpha}$ be the roots.

$\therefore \alpha + \frac{1}{\alpha} = \frac{-b}{a}$ ... (1)

and $\alpha \cdot \frac{1}{\alpha} = \frac{c}{a} \quad \Rightarrow 1 = \frac{c}{a}$

$\Rightarrow \quad \boxed{c = a}$

which is the required condition.

7. If the equations $x^2 - ax + b = 0$ and $x^2 - ex + f = 0$ have one root in common and if the second equation has equal roots, then prove that $ae = 2(b + f)$.

Sol : Given equations are $x^2 - ax + b = 0$ ...(1)

and $x^2 - ex + f = 0$ ...(2)

Let α be the common root.

Let α, β be the roots of $x^2 - ax + b = 0$

Then $\alpha + \beta = a$ and $\alpha\beta = b$ ...(3)

Let α, α be the roots of $x^2 - ex + f = 0$

[$\because$ the roots are equal]

$\therefore \alpha + \alpha = e, \quad \alpha \times \alpha = f$

$\Rightarrow \quad 2\alpha = e$, and $\alpha^2 = f$ ...(4)

Now LHS $= ae = (\alpha + \beta)\, 2\alpha$

LHS $= ae = 2\alpha^2 + 2\alpha\beta$

$= 2(f) + 2b$ [From (4)]

$= 2(b + f) = $ RHS.

Hence proved.

8. Discuss the nature of roots of (i) $-x^2 + 3x + 1 = 0$, (ii) $4x^2 - x - 2 = 0$, (iii) $9x^2 + 5x = 0$.

Sol : **(i) $-x^2 + 3x + 1 = 0$**

Given equation is $-x^2 + 3x + 1 = 0$

Here $a = -1, b = 3, c = 1$

$\therefore D = b^2 - 4ac = 3^2 - 4(-1)(1)$

$= 9 + 4 = 13$

Since $D > 0$, the roots are real and distinct.

(ii) $4x^2 - x - 2 = 0$

Here $a = 4, \; b = -1, c = -2$

$\therefore D = b^2 - 4ac = (-1)^2 - 4(4)(-2)$

$= 1 + 32 = 33$

Since $D > 0$, the roots are real and distinct

(iii) $9x^2 + 5x = 0$

Here $a = 9, b = 5, c = 0$

$\therefore D = b^2 - 4ac = 5^2 - 4(9)(0) = 25$

$D > 0$ and it is a perfect square, the roots are real and distinct.

9. Without sketching the graphs, find whether the graphs of the following functions will intersect the x-axis and if so in how many points.

(i) $y = x^2 + x + 2$, (ii) $y = x^2 - 3x - 7$, (iii) $y = x^2 + 6x + 9$.

Sol : **(i) $y = x^2 + x + 2$**

Here $a = 1, b = 1, c = 2$

$D = b^2 - 4ac = (1)^2 - 4(1)(2) = 1 - 8 = -7$

Since $D < 0$, the given curve will lie above the X-axis.

Hence the given graph will not intersect the X-axis.

(ii) $y = x^2 - 3x - 7$

Here $a = 1, \; b = -3, \; c = -7$

$\therefore D = b^2 - 4ac = (-3)^2 - 4(1)(-7)$

$= 9 + 28 = 37 > 0$

Since the roots are real and distinct the parabola intersects the X-axis at two different points.

(iii) $y = x^2 + 6x + 9$

Here $a = 1, \; b = 6, \; c = 9$

$\therefore D = b^2 - 4ac = (6)^2 - 4(1)(9)$

$= 36 - 36 = 0$

Since $D = 0$, the parabola touches the X -axis at only one point.

10. Write $f(x) = x^2 + 5x + 4$ in completed square form.

Sol : Let $y = x^2 + 5x + 4$

$= x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 4 = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 4$

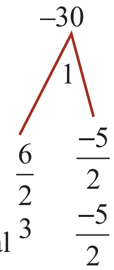
$= \left(x + \frac{5}{2}\right)^2 + \frac{-25+16}{4} = \left(x + \frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2$

EXERCISE 2.5

1. Solve $2x^2 + x - 15 \le 0$. **[Hy. - 2023; July-'24]**

Sol : Given inequality is $2x^2 + x - 15 \le 0$.

On factorising we get,

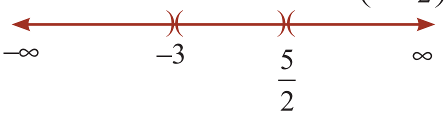


$(x+3)(2x-5) \le 0 \Rightarrow x = -3;\ x = \frac{5}{2}$

(i.e.) The critical points are $-3, \frac{5}{2}$, where the factors vanish.

Draw the number line and mark the critical points.

The possible intervals are $(-\infty, -3)$, $\left(-3, \frac{5}{2}\right)$ $\left(\frac{5}{2}, \infty\right)$



Interval	Sign of $(x+3)$	Sign of $(2x-5)$	Sign of $2x^2+x-15$
$(-\infty, -3)$ [say $x = -4$]	–	–	+
$\left(-3, \frac{5}{2}\right)$ [say $x = 0$]	+	–	–
$\left(\frac{5}{2}, \infty\right)$ [say $x = 3$]	+	+	+

The inequality $2x^2 + x - 15 \le 0$ is satisfied only in the interval $\left[-3, \frac{5}{2}\right]$.

$\therefore$ Solution set is $\left[-3, \frac{5}{2}\right]$.

2. Solve $-x^2 + 3x - 2 \ge 0$.

Sol : Given inequality is $-x^2 + 3x - 2 \ge 0$.

$\Rightarrow x^2 - 3x + 2 \le 0$ $\quad [\because a \ge b \Rightarrow -a \le -b]$.

$\Rightarrow (x-1)(x-2) \le 0$.

The critical points are 1 and 2 and the possible intervals are $(-\infty, 1)$ $(1, 2)$ and $(2, \infty)$

Intervals	Sign of $(x-1)$	Sign of $(x-2)$	Sign of x^2-3x+2
$(-\infty, 1)$ (say $x = 0$)	–	–	+
$(1, 2)$ (say $x = 1.5$)	+	–	–
$(2, \infty)$ (say $x = 3$)	+	+	+

The inequality $x^2 - 3x + 2 < 0$ is satisfied only in the interval [1, 2].

$\therefore$ Solution set is [1, 2].

EXERCISE 2.6

1. Find the zeros of the polynomial function $f(x) = 4x^2 - 25$.

Sol : Given $f(x) = 4x^2 - 25$

To find the zeros of $f(x)$, put $f(x) = 0$

$\Rightarrow\ 4x^2 - 25 = 0 \Rightarrow 4x^2 = 25$

$\Rightarrow\ x^2 = \frac{25}{4} \Rightarrow x = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$

Hence $\frac{5}{2}, -\frac{5}{2}$ are the zeros of $f(x)$.

2. If $x = -2$ is one root of $x^3 - x^2 - 17x = 22$, then find the other roots of the equation.

Sol : Given equation is $x^3 - x^2 - 17x = 22$

$\Rightarrow\ x^3 - x^2 - 17x - 22 = 0$

Since $x = -2$ is one root of the equation, $(x + 2)$ is a divisor of $x^3 - x^2 - 17x - 22 = 0$

$\therefore$ Using synthetic division,

–2	1	– 1	–17	–22
	0	–2	6	22
	1	–3	–11	0

$\therefore$ Consider $x^2 - 3x - 11$

Here $a = 1, b = -3, c = -11$

$$\frac{3 \pm \sqrt{(-3)^2 - 4(1)(-11)}}{2} = \frac{3 \pm \sqrt{9+44}}{2} = \frac{3 \pm \sqrt{53}}{2}$$

Hence the other roots are $\frac{3+\sqrt{53}}{2}, \frac{3-\sqrt{53}}{2}$

3. Find the real roots of $x^4 = 16$

Sol : Given equation is $x^4 = 16$

$\Rightarrow\ x^4 - 16 = 0 \Rightarrow (x^2)^2 - (4)^2 = 0$

$\Rightarrow\ (x^2 + 4)(x^2 - 4) = 0$ $\quad [\because a^2 - b^2 = (a+b)(a-b)]$

$\Rightarrow\ x^2 = -4, x^2 = 4$

$\Rightarrow$ When $x^2 = -4$,

$x = \sqrt{-4}$ No Real roots.

when $x^2 = 4$,

$x = \sqrt{4} \Rightarrow x = \pm 2$

$\therefore$ Required real roots are +2, –2.

4. Solve: $(2x+1)^2 - (3x+2)^2 = 0$

Sol : Given equation is $(2x+1)^2 - (3x+2)^2 = 0$

$\Rightarrow\ (2x + 1 + 3x + 2)(2x + 1 - 3x - 2) = 0$ $\quad [\because a^2 - b^2 = (a+b)(a-b)]$

$\Rightarrow\ (5x+3)(-x-1) = 0$

$\Rightarrow\ (5x+3)(x+1) = 0$

$\Rightarrow x = \frac{-3}{5}$ or -1 $\therefore x = -\frac{3}{5}$ (or) $x = -1$

Exercise 2.7

1. Factorize: $x^4 + 1$ (Hint: Try completing the square.)

Sol : Given equation is $x^4 + 1$

$$= x^2\left[x^2+\frac{1}{x^2}\right] = x^2\left[\left(x+\frac{1}{x}\right)^2-2\right]$$

[By completing the square]

$$= x^2\left[\left(x+\frac{1}{2}\right)^2-(\sqrt{2})^2\right]$$

$$= x^2\left[\left(x+\frac{1}{x}+\sqrt{2}\right)\left(x+\frac{1}{x}-\sqrt{2}\right)\right]$$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$$= x^2\left[\left(\frac{x^2+1+\sqrt{2}x}{x}\right)\left(\frac{x^2+1-\sqrt{2}x}{x}\right)\right]$$

$$= \cancel{x^2}\left[\frac{(x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x)}{\cancel{x^2}}\right]$$

$\therefore x^4 + 1 = (x^2 + \sqrt{2}\,x + 1)(x^2 - \sqrt{2}\,x + 1)$

2. If $x^2 + x + 1$ is a factor of the polynomial $3x^3 + 8x^2 + 8x + a$, then find the value of a.

Sol : Let $f(x) = 3x^3 + 8x^2 + 8x + a$

Since $(x^2 + x + 1)$ is a factor of $f(x)$, $f(x)$ is divisible by $x^2 + x + 1$

```
                 3x + 5
           ┌─────────────────────
x² + x + 1 │ 3x³ + 8x² + 8x + a
             (−)   (−)   (−)
             3x³ + 3x² + 3x
           ──────────────────────
                   5x² + 5x + a
                   (−)   (−)  (−)
                   5x² + 5x + 5
           ──────────────────────
                         a − 5
```

Since $f(x)$ is divisible by $x^2 + x + 1$, the remainder is zero.

$\therefore a - 5 = 0 \Rightarrow a = 5$

Exercise 2.8

1. Find all values of x for which $\dfrac{x^3(x-1)}{(x-2)} > 0$.

Sol : Given inequality is $\dfrac{x^3(x-1)}{x-2} > 0$

The critical numbers are 0, 1, 2

The possible intervals are $(-\infty, 0)$ $(0, 1)$ $(1, 2)$ $(2, \infty)$

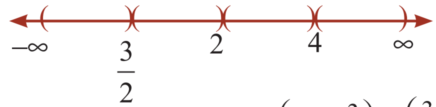
Interval	Sign of x^3	Sign of $(x-1)$	Sign of $(x-2)$	Sign of $\dfrac{x^3(x-1)}{x-2}$
$(-\infty, 0)$ Say $x = -1$	–	–	–	–
$(0, 1)$ Say $x = \frac{1}{2}$	+	–	–	+
$(1, 2)$ Say $x = 1\frac{1}{2} = \frac{3}{2}$	+	+	–	–
$(2, \infty)$ Say $x = 3$	+	+	+	+

The given inequality $\dfrac{x^3(x-1)}{x-2} > 0$ is satisfied by the intervals $(0, 1)$ and $(2, \infty)$. $\therefore$ Solution set is $(0, 1) \cup (2, \infty)$

2. Find all values of x that satisfies the inequality $\dfrac{2x-3}{(x-2)(x-4)} < 0$. **[Mar. - 2020; Qy. - 2019]**

Sol : Given inequality is $\dfrac{2x-3}{(x-2)(x-4)} < 0$.

The critical numbers are $x = \frac{3}{2}, 2, 4$



$\therefore$ The possible intervals are $\left(-\infty, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 2\right)$, $(2, 4)$ and $(4, \infty)$.

Intervals	Sign of $2x-3$	Sign of $x-2$	Sign of $x-4$	Sign of $\dfrac{2x-3}{(x-2)(x-4)}$
$\left(-\infty, \frac{3}{2}\right)$ Say $x = 0$	–	–	–	–
$\left(\frac{3}{2}, 2\right)$ Say $x = \frac{7}{4}$	+	–	–	+
$(2, 4)$ Say $x = 3$	+	+	–	–
$(4, \infty)$ Say $x = 5$	+	+	+	+

$\dfrac{2x-3}{(x-2)(x-4)} < 0$ in $\left(-\infty, \frac{3}{2}\right) \cup (2, 4)$

3. Solve: $\dfrac{x^2-4}{x^2-2x-15} \leq 0.$ **[Hy. - 2019]**

Sol : Given inequality is $\dfrac{x^2-4}{x^2-2x-15} \leq 0$

$$\Rightarrow \frac{(x+2)(x-2)}{(x-5)(x+3)} \leq 0$$

The critical numbers are –2, 2, 5, –3

∴ The possible intervals are (–∞, –3) (– 3, –2) (–2, 2) (2, 5) and (5, ∞)

Interval	Sign of $(x+2)$	Sign of $(x-2)$	Sign of $(x-5)$	Sign of $(x+3)$	Sign of $\dfrac{(x+2)(x-2)}{(x-5)(x+3)}$
(– ∞,–3) Say $x = -5$	–	–	–	–	+
(–3, –2) Say $x = -2.5$	–	–	–	+	–
(–2, 2) Say $x = 0$	+	–	–	+	+
(2, 5) Say $x = 3$	+	+	–	+	–
(5, ∞) Say $x = 6$	+	+	+	+	+

The inequality $\dfrac{(x+2)(x-2)}{(x-5)(x+3)} \leq 0$ is satisfied by the intervals (–3, –2) and (2,5) and it can be written as (–3, 2] and [2,5)

∴ Solution Set is (–3, –2] ∪ [2, 5).

Exercise 2.9

Resolve the following rational expressions into partial fractions.

1. $\dfrac{1}{x^2-a^2}$ **[Mar. - 2020; Sep. - 2021]**

2. $\dfrac{3x+1}{(x-2)(x+1)}$

3. $\dfrac{x}{(x^2+1)(x-1)(x+2)}$

4. $\dfrac{x}{(x-1)^3}$ **[QY-'24]**

5. $\dfrac{1}{x^4-1}$

6. $\dfrac{(x-1)^2}{x^3+x}$

7. $\dfrac{x^2+x+1}{x^2-5x+6}$

8. $\dfrac{x^3+2x+1}{x^2+5x+6}$

9. $\dfrac{x+12}{(x+1)^2(x-2)}$

10. $\dfrac{6x^2-x+1}{x^3+x^2+x+1}$

11. $\dfrac{2x^2+5x-11}{x^2+2x-3}$

12. $\dfrac{7+x}{(1+x)(1+x^2)}$ **[July-'24]**

Sol : **1.** $\dfrac{1}{x^2-a^2}$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)}$$

$$\text{Let } \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$\Rightarrow \quad 1 = A(x+a) + B(x-a) \quad \ldots(1)$$

Putting $x = a$ in (1) we get,

$$1 = A(2a) \Rightarrow A = \frac{1}{2a}$$

Putting $x = -a$ in (1) we get,

$$1 = B(-2a) \Rightarrow B = -\frac{1}{2a}$$

$$\therefore \frac{1}{x^2-a^2} = \frac{\frac{1}{2a}}{x-a} - \frac{\frac{1}{2a}}{x+a}$$

$$= \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}$$

2. $\dfrac{3x+1}{(x-2)(x+1)}$ **[June - 2019; Aug. - 2022]**

$$\frac{3x+1}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

$$\Rightarrow \frac{3x+1}{(x-2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$\Rightarrow \quad 3x+1 = A(x+1) + B(x-2) \ldots(1)$$

Putting $x = -1$ in (1) we get,

$$-3+1 = B(-3)$$

$$\Rightarrow \quad -2 = -3B \quad \Rightarrow \boxed{B = \frac{2}{3}}$$

Putting $x = 2$ in (1) we get,

$$6+1 = A(2+1)$$

$$\Rightarrow \quad 7 = 3A \Rightarrow \boxed{\frac{7}{3} = A}$$

$$\therefore \frac{3x+1}{(x-2)(x+1)} = \frac{\frac{7}{3}}{x-2} + \frac{\frac{2}{3}}{x+1}$$

$$= \frac{7}{3(x-2)} + \frac{2}{3(x+1)}$$

3. $\dfrac{x}{(x^2+1)(x-1)(x+2)}$

$$\frac{x}{(x^2+1)(x-1)(x+2)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x+2)}$$

$$\Rightarrow \frac{x}{(x^2+1)(x-1)(x+2)}$$

$$= \frac{(Ax+B)(x-1)(x+2)+C(x^2+1)(x+2)+D(x^2+1)(x-1)}{(x^2+1)(x-1)(x+2)}$$

$$\Rightarrow \quad x = (Ax+B)(x-1)(x+2) + C(x^2+1)(x+2) + D(x^2+1)(x-1) \quad \ldots(1)$$

Putting $x = 1$ in (1) we get,

$$1 = C(2)(3) \Rightarrow \boxed{C = \frac{1}{6}}$$

Putting $x = -2$ in (1) we get,

$$-2 = D(5)(-3) \Rightarrow \boxed{D = \frac{2}{15}}$$

Putting $x = 0$ in (1) we get,

$$0 = B(-1)(2) + C(1)(2) + D(1)(-1)$$

$$\Rightarrow 0 = -2B + 2C - D$$

$$\Rightarrow 0 = -2B + 2\left(\frac{1}{6}\right) - \frac{2}{15} \text{ [substituting the values of C and D]}$$

$$\Rightarrow 2B = \frac{2}{6} - \frac{2}{15} = \frac{1}{3} - \frac{2}{15}$$

$$\Rightarrow 2B = \frac{5-2}{15} = \frac{3}{15} = \frac{1}{5}$$

$$\Rightarrow \boxed{B = \frac{1}{10}}$$

Equating the Co-efficient of x^3 in (1) we get,

$$0 = A + C + D$$

$$\Rightarrow A = -C - D \quad \Rightarrow A = -\frac{1}{6} - \frac{2}{15}$$

$$\Rightarrow A = \frac{-5-4}{30} = \frac{-9}{30} = \frac{-3}{10}$$

$$\Rightarrow \boxed{A = \frac{-3}{10}}$$

$$\therefore \frac{x}{(x^2+1)(x-1)(x+2)} = \frac{\frac{-3}{10}x + \frac{1}{10}}{x^2+1} + \frac{\frac{1}{6}}{x-1} - \frac{\frac{2}{15}}{x+2}$$

$$= \frac{-3x+1}{10(x^2+1)} + \frac{1}{6(x-1)} + \frac{2}{15(x+2)} \text{ (or)}$$

$$\frac{1-3x}{10(x^2+1)} + \frac{1}{6(x-1)} + \frac{2}{15(x+2)}$$

4. $\dfrac{x}{(x-1)^3}$

Sol :

$$\frac{x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$\Rightarrow \frac{x}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$\Rightarrow x = A(x-1)^2 + B(x-1) + C \quad ...(1)$$

Putting $x = 1$ in (1) we get

$$\boxed{1 = C}$$

Putting $x = 0$ in (1) we get

$$0 = A - B + C$$

$$0 = A - B + 1$$

$$\Rightarrow A - B = -1 \quad ...(2)$$

Equating the co-efficient of x^2 we get

$$\boxed{0 = A}$$

Substituting A = 0 in (2) we get,

$$0 - B = -1$$

$$\Rightarrow \boxed{B = 1}$$

$$\therefore \frac{x}{(x-1)^3} = \frac{0}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$= \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$$

5. $\dfrac{1}{x^4-1}$

Sol :

$$\frac{1}{x^4-1} = \frac{1}{(x^2+1)(x^2-1)}$$

$$= \frac{1}{(x^2+1)(x+1)(x-1)}$$

$$\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$\Rightarrow \frac{1}{x^4-1} = \frac{(Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)}{(x^2+1)(x^2-1)}$$

$$\Rightarrow 1 = (Ax+B)(x^2-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1) \quad ...(1)$$

Putting $x = 1$ in (1) we get,

$$1 = D(2)(2)$$

$$\Rightarrow \boxed{D = \frac{1}{4}}$$

Putting $x = -1$ in (1) we get,

$$1 = C(2)(-2)$$

$$\Rightarrow \boxed{C = -\frac{1}{4}}$$

Equating the co-efficient of x^3 we get

$$0 = A + C + D$$

$$\Rightarrow A = -C - D$$

$$\Rightarrow A = \frac{1}{4} - \frac{1}{4} = 0 \Rightarrow \boxed{A = 0}$$

Putting $x = 0$ in (1) we get,

$$1 = -B - C + D$$

$$\Rightarrow 1 = -B + \frac{1}{4} + \frac{1}{4}$$

$$\Rightarrow B = -1 + \frac{1}{2} \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$\therefore \frac{1}{x^4-1} = \frac{0x - \frac{1}{2}}{x^2+1} + \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1}$$

$$\Rightarrow \frac{1}{x^4-1} = \frac{-\frac{1}{2}}{x^2+1} - \frac{\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1}$$

$$= -\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)} = -\frac{1}{2(x^2-1)} - \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right)$$

$$= \frac{-1}{2(x^2+1)} - \frac{1}{4}\left(\frac{\cancel{x}-1-\cancel{x}-1}{x^2-1}\right) = \frac{-1}{2(x^2+1)} - \frac{1}{\cancel{4}_{2}}\left(\frac{\cancel{-2}^{1}}{x^2-1}\right)$$

$$= \frac{-1}{2(x^2+1)} + \frac{1}{2(x^2-1)} = \frac{-2}{2(x^2+1)} + \frac{1}{2(x^2-1)}$$

$$\left[\text{Here } \frac{-1}{2(x^2+1)} = \frac{-2}{2(x^2+1)}\right] (\text{or}) = \frac{1}{2(x^2-1)} - \frac{2}{2(x^2+1)}$$

6. $\dfrac{(x-1)^2}{x^3+x}$

Sol : $\dfrac{(x-1)^2}{x^3+x} = \dfrac{(x-1)^2}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$\Rightarrow (x-1)^2 = A(x^2+1) + (Bx+C)(x)$ ...(1)

Putting $x = 0$ in (1) we get,

$\boxed{1 = A}$

Putting $x = 1$ in (1) we get,

$0 = A(2) + (B + C)(1)$

$\Rightarrow 0 = 2A + B + C$ ...(2)

Equating the Co-efficient of x^2 in (1) we get,

$1 = A + B$

$\Rightarrow 1 = 1 + B \Rightarrow \boxed{B = 0}$

Substituting A = 1, B = 0 in (2) we get,

$0 = 2 + 0 + C$

$\Rightarrow \boxed{C = -2}$

$\therefore \dfrac{(x-1)^2}{x^3+x} = \dfrac{1}{x} + \dfrac{0x-2}{x^2+1}$

$\Rightarrow \dfrac{(x-1)^2}{x^3+x} = \dfrac{1}{x} - \dfrac{2}{x^2+1}$

7. $\dfrac{x^2+x+1}{x^2-5x+6}$ **[Qy. - 2019]**

Sol : Since the degree of the numerator is equal to the degree of the denominator, let us divide the numerator by the denominator.

$$\begin{array}{r|l} & 1 \\ \hline x^2-5x+6 & \cancel{x^2}+x+1 \\ & \underset{(-)}{\cancel{x^2}}\ \underset{(+)}{-5x}\ \underset{(-)}{+6} \\ \hline & 6x-5 \end{array}$$

$\therefore \dfrac{x^2+x+1}{x^2-5x+6} = 1 + \dfrac{6x-5}{x^2-5x+6}$ ...(1)

Consider $\dfrac{6x-5}{x^2-5x+6} = \dfrac{6x-5}{(x-3)(x-2)}$

$= \dfrac{A}{(x-3)} + \dfrac{B}{(x-2)}$

$\Rightarrow \dfrac{6x-5}{x^2-5x+6} = \dfrac{A(x-2)+B(x-3)}{(x-3)(x-2)}$

$\Rightarrow 6x - 5 = A(x-2) + B(x-3)$ ...(2)

Putting $x = 2$ in (2) we get,

$7 = B(-1) \Rightarrow \boxed{B = -7}$

Putting $x = 3$ in (2) we get,

$13 = A(1) \Rightarrow \boxed{A = 13}$

$\therefore \dfrac{6x-5}{x^2-5x+6} = \dfrac{13}{x-3} - \dfrac{7}{x-2}$ ... (3)

Substituting (3) in (1) we get,

$\dfrac{x^2+x+1}{x^2-5x+6} = 1 + \dfrac{13}{x-3} - \dfrac{7}{x-2}$

8. $\dfrac{x^3+2x+1}{x^2+5x+6}$

Sol : Since the numerator's degree is more than the denominator's degree, let us divide the numerator by the denominator.

$$\begin{array}{r|l} & x-5 \\ \hline x^2+5x+6 & \cancel{x^3}+0x^2+2x+1 \\ & \underset{(-)}{\cancel{x^3}}\ \underset{(-)}{+5x^2}\ \underset{(-)}{+6x} \\ \hline & -\cancel{5x^2}-4x+1 \\ & \underset{(+)}{-\cancel{5x^2}}\ \underset{(+)}{-25x}\ \underset{(+)}{-30} \\ \hline & 21x+31 \end{array}$$

$\therefore \dfrac{x^3+2x+1}{x^2+5x+6} = (x-5) + \dfrac{21x+31}{x^2+5x+6}$ ...(1)

Consider $\dfrac{21x+31}{x^2+5x+6} = \dfrac{21x+31}{(x+3)(x+2)}$

$= \dfrac{A}{x+3} + \dfrac{B}{x+2}$ ...(2)

$21x + 31 = A(x+2) + B(x+3)$ ... (3)

Putting $x = -2$ in (3) we get,

$-11 = B(1)$

$\Rightarrow \boxed{B = -11}$

Putting $x = -3$ in (3) we get,

$-32 = A(-1)$

$\Rightarrow \boxed{A = 32}$

$\therefore \dfrac{21x+31}{x^2+5x+6} = \dfrac{32}{x+3} - \dfrac{11}{x+2}$

Substituting this in (1) we get,

$\dfrac{x^3+2x+1}{x^2+5x+6} = (x-5) + \dfrac{32}{x+3} - \dfrac{11}{x+2}$

9. $\dfrac{x+12}{(x+1)^2(x-2)}$ **[Hy. - 2023]**

Sol : $\dfrac{x+12}{(x+1)^2(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-2}$

$\Rightarrow \quad x + 12 = A(x+1)(x-2) + B(x-2) + C(x+1)^2 \quad ...(1)$

Putting $x = -1$ in (1) we get,

$11 = B(-3)$

$\Rightarrow \quad \boxed{B = \dfrac{-11}{3}}$

Putting $x = 2$ in (1) we get,

$14 = C(9)$

$\Rightarrow \quad \boxed{C = \dfrac{14}{9}}$

Equating the Co-efficient of x^2 in (1) we get,

$0 = A + C \Rightarrow A = -C \Rightarrow \boxed{A = \dfrac{-14}{9}}$

$\therefore \dfrac{x+12}{(x+1)^2(x-2)} = \dfrac{\frac{-14}{9}}{x+1} - \dfrac{\frac{11}{3}}{(x+1)^2} + \dfrac{\frac{14}{9}}{x-2}$

$= -\dfrac{14}{9(x+1)} - \dfrac{11}{3(x+1)^2} + \dfrac{14}{9(x-2)}$

10. $\dfrac{6x^2-x+1}{x^3+x^2+x+1}$

Sol : $\dfrac{6x^2-x+1}{(x^3+1)(x+1)} = \dfrac{6x^2-x+1}{x^2(x+1)+1(x+1)}$

$= \dfrac{6x^2-x+1}{(x^2+1)(x+1)}$

$\Rightarrow \quad \dfrac{6x^2-x+1}{(x+1)(x^2+1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+1}$

$\Rightarrow \quad 6x^2 - x + 1 = (Ax+B)(x+1) + C(x^2+1) \quad ...(1)$

Putting $x = -1$ in (1) we get,

$6 + 1 + 1 = C(1+1) \quad \Rightarrow 8 = 2C$

$\Rightarrow \quad \boxed{C = 4}$

Equating the co-efficient of x^2 in (1) we get,

$6 = A + C \quad \Rightarrow 6 = A + 4$

$\Rightarrow \quad A = 6 - 4$

$\Rightarrow \quad \boxed{A = 2}$

Putting $x = 0$ in (1) we get,

$1 = B + C \quad \Rightarrow 1 = B + 4$

$\Rightarrow \quad 1 - 4 = B$

$\Rightarrow \quad \boxed{B = -3}$

$\therefore \dfrac{6x^2-x+1}{x^3+x^2+x+1} = \dfrac{2x-3}{x^2+1} + \dfrac{4}{x+1}$

11. $\dfrac{2x^2+5x-11}{x^2+2x-3}$ **[First Mid - 2018]**

Sol : Since the numerator's degree is equal to the denominator, let us divide the numerator by the denominator.

$$\begin{array}{r|l} & \quad 2 \\ x^2+2x-3 & 2x^2+5x-11 \\ & \underset{(-)}{2x^2} \underset{(-)}{+4x} \underset{(+)}{-6} \\ \hline & \qquad x-5 \end{array}$$

$\therefore \dfrac{2x^2+5x-11}{x^2+2x-3} = 2 + \dfrac{x-5}{x^2+2x-3} \quad ...(1)$

Consider $\dfrac{x-5}{x^2+2x-3} = \dfrac{x-5}{(x+3)(x-1)} = \dfrac{A}{x+3} + \dfrac{B}{x-1}$

$= \dfrac{A(x-1)+B(x+3)}{(x+3)(x-1)}$

$\therefore x - 5 = A(x-1) + B(x+3) \quad ...(2)$

Putting $x = 1$ in (2) we get,

$-4 = B(4)$

$\Rightarrow \quad \boxed{B = -1}$

Putting $x = -3$ in (2) we get,

$-8 = A(-4)$

$\Rightarrow \quad \boxed{A = 2}$

-3: 3, -1; $(x+3)$ $(x-1)$

$\therefore \dfrac{x-5}{x^2+2x-3} = \dfrac{2}{x+3} - \dfrac{1}{x-1}$

Substituting this in (1) we get,

$\dfrac{2x^2+5x-11}{x^2+2x-3} = 2 + \left(\dfrac{2}{x+3} - \dfrac{1}{x-1}\right)$

12. $\dfrac{7+x}{(1+x)(1+x^2)}$ **[Qy. - 2018]**

Sol : $\dfrac{7+x}{(1+x)(1+x^2)} = \dfrac{A}{1+x} + \dfrac{Bx+C}{x^2+1}$

$\Rightarrow \quad x + 7 = A(x^2+1) + (Bx+C)(1+x) \quad ...(1)$

Putting $x = -1$ in (1) we get,

$6 = A(2)$

$\Rightarrow \quad \boxed{A = 3}$

Equating the co-efficient of x^2 in (1) we get,

$0 = A + B$

$\Rightarrow \quad 0 = 3 + B$

$\Rightarrow \quad \boxed{B = -3}$

Putting $x = 0$ in (1) we get,

$7 = A + C$

$\Rightarrow \quad 7 = 3 + C$

$\Rightarrow \quad \boxed{C = 4}$

$$\therefore \frac{7+x}{(1+x)(1+x^2)} = \frac{A}{1+x}+\frac{Bx+C}{x^2+1}$$

$$= \frac{3}{1+x}+\frac{(-3x+4)}{x^2+1}$$

Exercise 2.10

Determine the region in the plane determined by the inequalities.

1. $x \le 3y, x \ge y.$

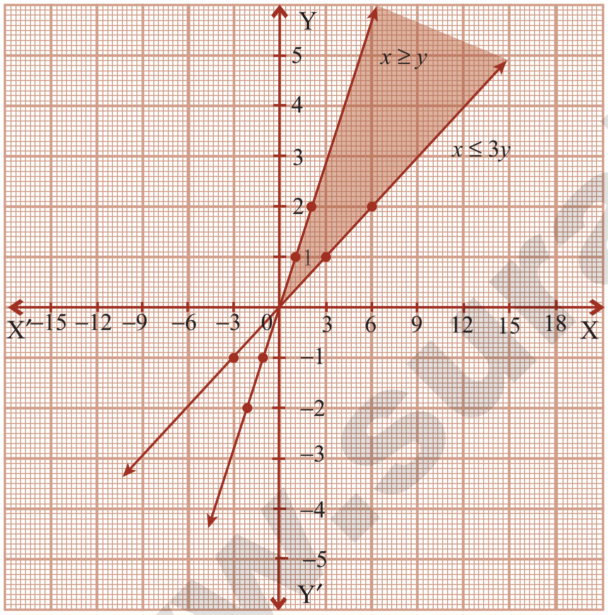
Sol : Given in equalities are $x \le 3y$, $x \ge y$

Suppose $x = 3y \Rightarrow \frac{x}{3} = y$

x	0	3	6	–3
y	0	1	2	–1

If $x = y$ then

x	1	2	–1	–2
y	1	2	–1	–2



$\therefore$ The shaded region is the solution for the inequalities.

2. $y \ge 2x, -2x + 3y \le 6.$

Sol : Suppose $y = 2x$

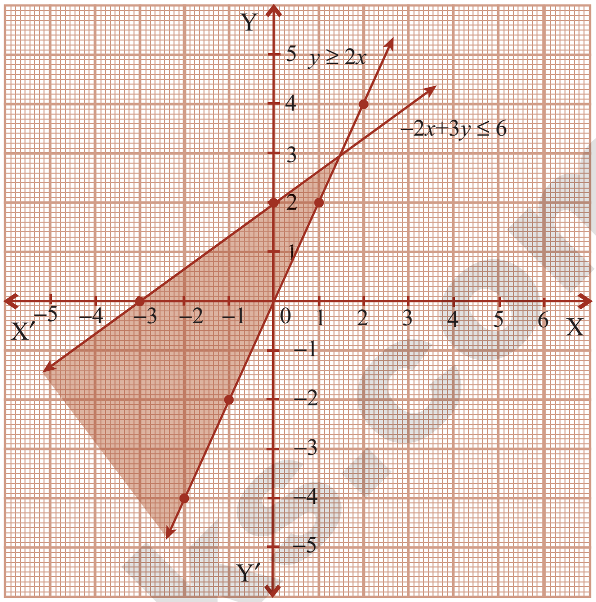
x	1	–1	2	–2
y	2	–2	4	–4

Then $-2x + 3y = 6$

$-2x = 6 - 3y$

$x = \frac{6-3y}{-2}$

x	0	–3
y	2	0

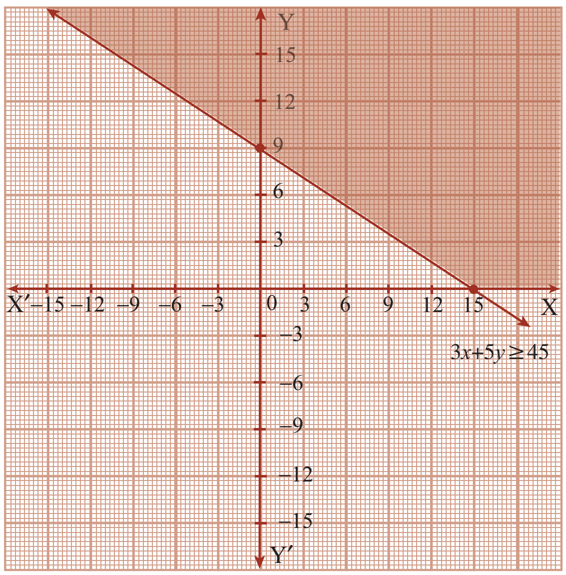


$\therefore$ The shaded region is the solution for the inequalities.

3. $3x + 5y \ge 45, x \ge 0, y \ge 0.$

Sol : If $3x + 5y = 45$

x	0	15
y	9	0



$x \ge 0$ is nothing but the positive portion of Y-axis and

$y \ge 0$ is the positive portion of X-axis.

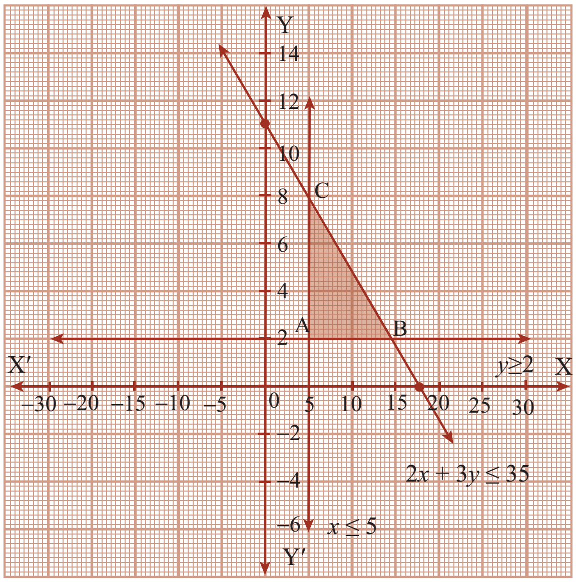
Shaded region is the required portion.

4. $\mathbf{2x + 3y \le 35, y \ge 2, x \ge 5.}$

Sol : If $2x + 3y = 35$ then

x	0	17.5
y	11.6	0

$y = 2$ is a line parallel to X-axis at a distance 2 units. $x = 5$ is a line parallel to Y-axis at a distance of 5 units.



ABC is the required shaded region.

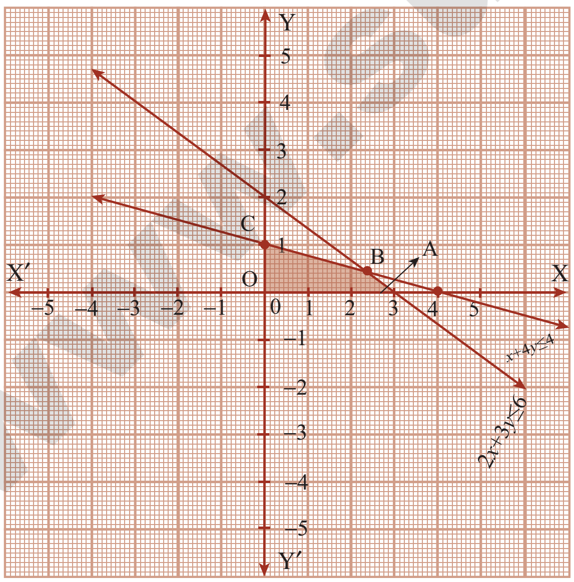
5. $\mathbf{2x + 3y \le 6, x + 4y \le 4, x \ge 0, y \ge 0.}$

Sol : If $2x + 3y = 6$ $\qquad$ $x + 4y = 4$

x	0	3
y	2	0

x	0	4
y	1	0

$x \ge 0, y \ge 0$ represents the area in the I quadrant.



OABC is the required shaded region.

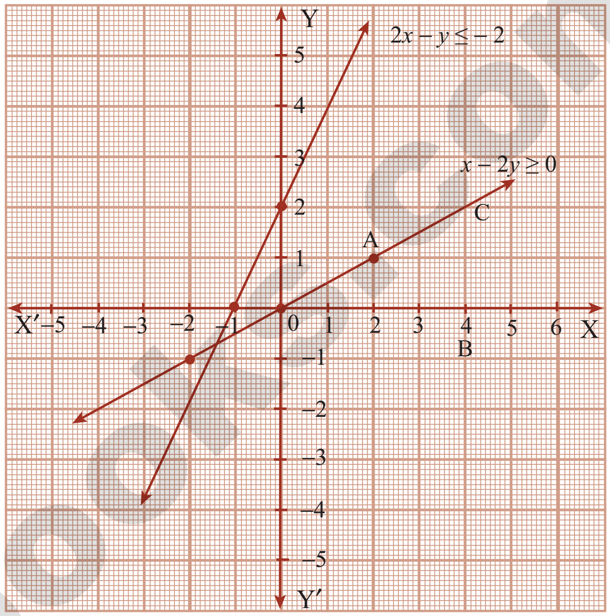
6. $\mathbf{x - 2y \ge 0, 2x - y \le -2, x \ge 0, y \ge 0.}$

Sol : If $x - 2y = 0$ $\qquad$ If $2x - y = -2$

$x = 2y$

x	0	2	–2
y	0	1	–1

x	0	–1
y	2	0



$x \ge 0, y \ge 0$ represents the portion in the I quadrant only Since there is no common part in the graph, it has no solution.

7. $\mathbf{2x + y \ge 8, x + 2y \ge 8, x + y \le 6.}$

Sol :

$2x + y = 8$

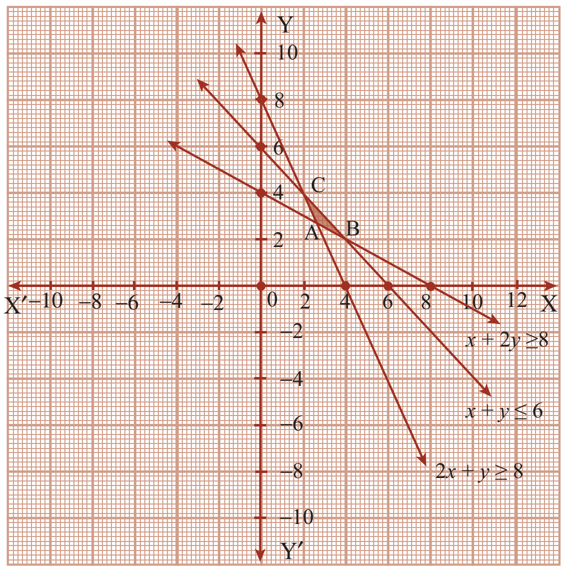
x	0	4
y	8	0

$x + 2y = 8$

x	0	8
y	4	0

$x + y = 6$

x	0	6
y	6	0



ABC is the required shaded region.

Exercise 2.11

1. Simplify:

(i) $(125)^{\frac{2}{3}}$ (ii) $16^{-\frac{3}{4}}$

(iii) $(-1000)^{\frac{-2}{3}}$ (iv) $(3^{-6})^{\frac{1}{3}}$

(v) $\dfrac{(27)^{\frac{-2}{3}}}{(27)^{\frac{-1}{3}}}$

Sol :

(i) $(125)^{\frac{2}{3}} = (5^3)^{\frac{2}{3}} = 5^{3\times\frac{2}{3}} = 5^2 = 25$

$\left[\because (a^m)^n = a^{mn}\right]$

(ii) $16^{-\frac{3}{4}} = (2^4)^{\frac{-3}{4}} = 2^{4\times\frac{-3}{4}} = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

(iii) $(-1000)^{\frac{-2}{3}} = \left[(-10)^3\right]^{\frac{-2}{3}} = (-10)^{-2} = \dfrac{1}{(-10)^2} = \dfrac{1}{100}$

(iv) $(3^{-6})^{\frac{1}{3}} = 3^{-6\times\frac{1}{3}} = 3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(v) $\dfrac{(27)^{\frac{-2}{3}}}{(27)^{\frac{-1}{3}}} = (27)^{\frac{-2}{3}+\frac{1}{3}} = (27)^{\frac{-1}{3}} = 3^{3\times\frac{-1}{3}} = 3^{-1} = \dfrac{1}{3}$

$\left[\because \dfrac{a^m}{a^n} = a^{m-n}\right]$

2. Evaluate $\left(\left[(256)^{\frac{-1}{2}}\right]^{\frac{-1}{4}}\right)^3$ **[Qy. - 2019]**

Solution : $\left(\left[(256)^{\frac{-1}{2}}\right]^{\frac{-1}{4}}\right)^3 = (256)^{\frac{-1}{2}\times\frac{-1}{4}\times 3}$

$[\because (a^m)^n = a^{mn}]$

$= (256)^{\frac{3}{8}} = (2^8)^{\frac{3}{8}} = 2^{8\times\frac{3}{8}} = 2^3 = 8$

3. If $\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 = \dfrac{9}{2}$**, then find the value of** $\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$ **for $x > 1$.** **[July-2023; QY-'24]**

Sol : Given $\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 = \dfrac{9}{2}$

$\Rightarrow x + \dfrac{1}{x} + 2x^{\frac{1}{2}}.\dfrac{1}{x^{\frac{1}{2}}} = \dfrac{9}{2} \Rightarrow x + \dfrac{1}{x} + 2 = \dfrac{9}{2}$

$\Rightarrow x + \dfrac{1}{x} = \dfrac{9}{2} - 2 = \dfrac{9-4}{2} = \dfrac{5}{2}$ ...(1)

Consider $\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2 = x + \dfrac{1}{x} - 2x^{\frac{1}{2}}.\dfrac{1}{x^{\frac{1}{2}}}$

$= x + \dfrac{1}{x} - 2$ [From (1)]

$\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2 = \dfrac{5}{2} - 2 = \dfrac{5-4}{2} = \dfrac{1}{2}$ [using 1]

$\therefore x^{\frac{1}{2}} - x^{-\frac{1}{2}} = \pm\dfrac{1}{\sqrt{2}}$

$\Rightarrow x^{\frac{1}{2}} - x^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2}}$ since $x > 1$.

4. Simplify and hence find the value of n : $\dfrac{3^{2n}9^2 3^{-n}}{3^{3n}} = 27$**.**

Sol : Given $\dfrac{3^{2n}9^2 3^{-n}}{3^{3n}} = 27$

$\Rightarrow \dfrac{3^{2n-n}.9^2}{3^{3n}} = 27$ $[\because a^m . a^n = a^{m+n}]$

$\Rightarrow \dfrac{3^n.(3^2)^2}{3^{3n}} = 27 \Rightarrow 3^{n-3n}\cdot(3^4) = 27$

$\left[\because \dfrac{a^m}{a^n} = a^{m-n} \text{ \& } (a^m)^n = a^{mn}\right]$

$\Rightarrow 3^{-2n}\cdot 3^4 = 27 \Rightarrow 3^{-2n+4} = 3^3.$

Equating the powers both sides we get, $-2n + 4 = 3$

$\Rightarrow -2n = 3 - 4 = -1 \Rightarrow 2n = 1$

$\Rightarrow n = \dfrac{1}{2}$

5. Find the radius of the spherical tank whose volume is $\dfrac{32\pi}{3}$ units.

Sol : Let r be the radius of the spherical tank.

Then, volume of the spherical tank = $\dfrac{32\pi}{3}$

$\Rightarrow \dfrac{4}{3}\pi r^3 = \dfrac{32\pi}{3} \Rightarrow 4r^3 = 32$

$\Rightarrow r^3 = \dfrac{32}{4} = 8 \Rightarrow r^3 = 2^3$

$\Rightarrow r = 2$

$\therefore$ Radius of the spherical tank is 2 units.

6. Simplify by rationalising the denominator $\dfrac{7+\sqrt{6}}{3-\sqrt{2}}$**.**

Sol : $\dfrac{7+\sqrt{6}}{3-\sqrt{2}} \times \dfrac{3+\sqrt{2}}{3+\sqrt{2}} \Rightarrow \dfrac{(7+\sqrt{6})(3+\sqrt{2})}{3^2-(\sqrt{2})^2}$

$\Rightarrow \dfrac{21+7\sqrt{2}+3\sqrt{6}+\sqrt{12}}{9-2} \Rightarrow \dfrac{21+7\sqrt{2}+3\sqrt{6}+2\sqrt{3}}{7}$

7. Simplify

$$\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}$$

Sol : Given **[Qy. - 2023; Mar. - 2024]**

$$\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2} \quad \ldots (1)$$

Multiplying each term by the conjugate of the denominator we get,

$$\frac{1}{3-\sqrt{8}}=\frac{1}{3-\sqrt{8}}\times\frac{3+\sqrt{8}}{3+\sqrt{8}}=\frac{3+\sqrt{8}}{3^2-\sqrt{8}^2}=\frac{3+\sqrt{8}}{9-8}=3+\sqrt{8}$$

$$\frac{1}{\sqrt{8}-\sqrt{7}}=\frac{1}{\sqrt{8}-\sqrt{7}}\times\frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}+\sqrt{7}}=\frac{\sqrt{8}+\sqrt{7}}{8-7}=\frac{\sqrt{8}+\sqrt{7}}{1}=\sqrt{8}+\sqrt{7}$$

$$\frac{1}{\sqrt{7}-\sqrt{6}}=\frac{1}{\sqrt{7}-\sqrt{6}}\times\frac{\sqrt{7}+\sqrt{6}}{\sqrt{7}+\sqrt{6}}=\frac{\sqrt{7}+\sqrt{6}}{\sqrt{7}^2-\sqrt{6}^2}=\frac{\sqrt{7}+\sqrt{6}}{7-6}=\sqrt{7}+\sqrt{6}$$

$$\frac{1}{\sqrt{6}-\sqrt{5}}=\frac{1}{\sqrt{6}-\sqrt{5}}\times\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}}=\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}^2-\sqrt{5}^2}=\frac{\sqrt{6}+\sqrt{5}}{6-5}$$

$$=\sqrt{6}+\sqrt{5}$$

$$\frac{1}{\sqrt{5}-2}=\frac{1}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{\sqrt{5}+2}{\sqrt{5}^2-2^2}=\frac{\sqrt{5}+2}{5-4}=\sqrt{5}+2$$

Substituting all these values in (1) we get,

$$\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}$$

$$=3+\sqrt{8}-\sqrt{8}-\sqrt{7}+\sqrt{7}+\sqrt{6}-\sqrt{6}-\sqrt{5}+\sqrt{5}+2$$

$$=5$$

8. If $x=\sqrt{2}+\sqrt{3}$ find $\dfrac{x^2+1}{x^2-2}$.

Sol :

$$\text{Given } x = \sqrt{2}+\sqrt{3}$$

$$\Rightarrow \quad x^2 = \left(\sqrt{2}+\sqrt{3}\right)^2$$

$$= 2+3+2\sqrt{6}=5+2\sqrt{6}$$

$$\therefore \quad \frac{x^2+1}{x^2-2} = \frac{5+2\sqrt{6}+1}{5+2\sqrt{6}-2}=\frac{6+2\sqrt{6}}{3+2\sqrt{6}}$$

$$\frac{6+2\sqrt{6}}{3+2\sqrt{6}}\times\frac{3-2\sqrt{6}}{3-2\sqrt{6}} = \frac{\left(6+2\sqrt{6}\right)\left(3-2\sqrt{6}\right)}{9-\left(2\sqrt{6}\right)^2}$$

$$=\frac{18-12\sqrt{6}+6\sqrt{6}-4\left(\sqrt{6}\right)^2}{9-24}=\frac{18-6\sqrt{6}-24}{-15}$$

$$=\frac{-6-6\sqrt{6}}{-15}=\frac{-3\left(2+2\sqrt{6}\right)}{-15}=\frac{2+2\sqrt{6}}{5}$$

Exercise 2.12

1. Let $b>0$ and $b\neq 1$. Express $y=b^x$ in logarithmic form. Also state the domain and range of the logarithmic function.

Sol : Given $y=b^x$, $b>0$, $b\neq 1$

Taking log base 'b' on both sides

$$\Rightarrow \quad \log_b y = \log_b b^x$$

$$\Rightarrow \quad \log_b y = x\log_b b \quad [\because \log m^n = n\log m]$$

$$\Rightarrow \quad \log_b y = x \quad [\because \log_b b = 1]$$

$$\therefore \text{ Domain for } x = \log_b y \text{ is } (0,\infty)$$

$$\therefore \text{ Range for } x = \log_b y \text{ is } (-\infty,\infty)$$

2. Compute $\log_9 27 - \log_{27} 9$ **[First Mid - 2018]**

Sol : Given $\log_9 27-\log_{27}9 = \log_9 3^3-\log_{27}3^2$

$$= 3\log_9 3 - 2\log_{27}3 \text{ [By power rule]}$$

$$=\frac{3}{\log_3 9}-\frac{2}{\log_3 27} \text{ [By change of base rule]}$$

$$=\frac{3}{\log_3 3^2}-\frac{2}{\log_3 3^3}=\frac{3}{2\log_3 3}-\frac{2}{3\log_3 3}$$

$$=\frac{3}{2}-\frac{2}{3}=\frac{9-4}{6}=\frac{5}{6} \quad [\because \log_3 3 = 1]$$

3. Solve : $\log_8 x+\log_4 x+\log_2 x = 11$.

Sol : Given $\log_8 x+\log_4 x+\log_2 x = 11$

$$\Rightarrow \quad \frac{1}{\log_x 8}+\frac{1}{\log_x 4}+\frac{1}{\log_x 2} = 11$$

$$\Rightarrow \quad \frac{1}{\log_x 2^3}+\frac{1}{\log_x 2^2}+\frac{1}{\log_x 2} = 11$$

$$\Rightarrow \quad \frac{1}{3\log_x 2}+\frac{1}{2\log_x 2}+\frac{1}{\log_x 2} = 11$$

$$\Rightarrow \quad \frac{1}{\log_x 2}\left(\frac{1}{3}+\frac{1}{2}+1\right) = 11$$

$$\Rightarrow \quad \frac{1}{\log_x 2}\left(\frac{2+3+6}{6}\right) = 11$$

$$\Rightarrow \quad \frac{1}{\log_x 2}\left(\frac{11}{6}\right) = 11$$

$$\Rightarrow \quad \frac{1}{\log_x 2} = 11\times\frac{6}{11}=6$$

$$\Rightarrow \quad \frac{1}{\log_x 2} = 6$$

$$\Rightarrow \quad \log_2 x = 6$$

$$\Rightarrow \quad 2^6 = x$$

$$\Rightarrow \quad x = 64$$

4. Solve : $\log_4 2^{8x} = 2^{\log_2 8}$

Sol :

$$8x \log_4 2 = 2^{\log_2 2^3}$$

$$\Rightarrow \quad 8x \log_4 2 = 2^{3\log_2 2}$$

$$\Rightarrow \quad 8x \log_4 2 = 2^{3(1)} \qquad [\because \log_2 2 = 1]$$

$$\Rightarrow \quad \frac{8x}{\log_2 4} = 8$$

$$\Rightarrow \quad x = \log_2 4 \Rightarrow x = \log_2 2^2$$

$$\Rightarrow \quad x = 2\log_2 2 \Rightarrow x = 2$$

5. If $a^2 + b^2 = 7ab$, Show that $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$. **[CRT - 2022]**

Sol : Given $a^2 + b^2 = 7ab$

Adding $2ab$ both sides we get,

$$a^2 + b^2 + 2ab = 7ab + 2ab$$

$$\Rightarrow \quad (a+b)^2 = 9ab \Rightarrow \frac{(a+b)^2}{9} = ab$$

$$\Rightarrow \quad \left(\frac{a+b}{3}\right)^2 = ab.$$

Taking square root, we get

$$\frac{a+b}{3} = \sqrt{ab}$$

Taking log on both sides, we get

$$\log\left(\frac{a+b}{3}\right) = \log(ab)^{\frac{1}{2}} = \frac{1}{2}\log(ab)$$

[By power rule]

$$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}[\log a + \log b]$$

[By product rule]

Hence proved.

6. Prove $\log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab} = 0$. **[July-'24]**

Sol :

$$\text{LHS} = \log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab}$$

$$= \log\left(\frac{a^2}{bc}\cdot\frac{b^2}{ca}\cdot\frac{c^2}{ab}\right)$$

[By product rule]

$$= \log\left(\frac{a^2b^2c^2}{a^2b^2c^2}\right) = \log 1 = 0$$

$= \text{RHS}$. Hence proved.

7. Prove that $\log_{10} 2 + 16\log_{10}\frac{16}{15} + 12\log_{10}\frac{25}{24} + 7\log_{10}\frac{81}{80} = 1$ **[Mar. - 2024]**

Sol :

$$\text{LHS} = \log_{10} 2 + 16\log_{10}\frac{16}{15} + 12\log_{10}\frac{25}{24} + 7\log_{10}\frac{81}{80}$$

$$= \log_{10} 2 + \log_{10}\left(\frac{16}{15}\right)^{16} + \log_{10}\left(\frac{25}{24}\right)^{12} + \log_{10}\left(\frac{81}{80}\right)^{7}$$

$$= \log_{10}\left[2 \times \frac{(2^4)^{16}}{(3\times5)^{16}} \times \frac{(5^2)^{12}}{(2^3\times3)^{12}} \times \frac{(3^4)^7}{(2^4\times5)^7}\right]$$

$$= \log_{10}\left[2^1 \times \frac{2^{64}}{3^{16}\times5^{16}} \times \frac{5^{24}}{2^{36}\times3^{12}} \times \frac{3^{28}}{2^{28}\times5^7}\right]$$

$$[\because (a^m)^n = a^{mn}]$$

$$= \log_{10}\frac{2^{1+64}\cdot5^{24}\cdot3^{28}}{3^{16+12}\cdot5^{16+7}\cdot2^{36+28}} \qquad \left[\because \frac{a^m}{a^n} = a^{m-n}\right]$$

$$= \log_{10}\frac{2^{65}\cdot5^{24}\cdot3^{28}}{3^{28}\cdot5^{23}\cdot2^{64}} = \log_{10}(2^{65-64}\times5^{24-23})$$

$$= \log_{10} 2^1 \times 5^1 = \log_{10} 10 = 1 = \text{RHS}.$$

Hence proved.

8. Prove $\log_{a^2} a \times \log_{b^2} b \times \log_{c^2} c = \frac{1}{8}$.

Sol :

$$\text{LHS} = \log_{a^2} a \times \log_{b^2} b \times \log_{c^2} c$$

$$= \frac{1}{\log_a a^2}\cdot\frac{1}{\log_b b^2}\cdot\frac{1}{\log_c c^2}$$

$$= \frac{1}{2\log_a a}\cdot\frac{1}{2\log_b b}\cdot\frac{1}{2\log_c c}$$

$$= \frac{1}{2(1)}\cdot\frac{1}{2(1)}\cdot\frac{1}{2(1)} \qquad [\because \log_a a = 1]$$

$$= \frac{1}{8} = \text{RHS. Hence proved.}$$

9. Prove $\log a + \log a^2 + \log a^3 + \ldots + \log a^n = \frac{n(n+1)}{2}\log a$ **[Govt. MQP - 2018; April - 2023]**

Sol :

$$\text{LHS} = \log a + \log a^2 + \log a^3 + \ldots + \log a^n$$

$$= \log a + 2\log a + 3\log a + \ldots + n\log a$$

$$= \log a\,(1 + 2 + 3 + \ldots.\; n)$$

$$= \log a\,\frac{(n)(n+1)}{2} \qquad \left[\because \sum n = \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{2}\log a = \text{RHS}$$

Hence proved.

10. **If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$, then prove that $xyz = 1$.** [Sep. - 2020; Qy. & First Mid - 2018]

Sol : Let $\frac{\log x}{y-z} = \frac{\log y}{z-x}=\frac{\log z}{x-y}=k$

$\Rightarrow \log x = k(y-z) = ky - kz$ ...(1)

$\log y = k(z-x) = kz - kx$ ...(2)

$\log z = k(x-y) = kx - ky$ ...(3)

Adding (1), (2) and (3)

$\log x + \log y + \log z = ky - kz + kz - kx + kx - ky = 0$

$\Rightarrow \log xyz = 0 = \log 1$

$\therefore xyz = 1$ Hence proved.

11. **Solve: $\log_2 x - 3\log_{\frac{1}{2}} x = 6$**

Sol : Given $\log_2 x - 3\log_{\frac{1}{2}} x = 6$

[Using change of base rule]

$\Rightarrow \frac{1}{\log_x 2}-\frac{3}{\log_x \frac{1}{2}} = 6$

$\Rightarrow \frac{1}{\log_x 2}-\frac{3}{\log_x 1-\log_x 2} = 6$

[Using quotient rule]

$\Rightarrow \frac{1}{\log_x 2}-\frac{3}{0-\log_x 2} = 6$

$\Rightarrow \frac{1}{\log_x 2}+\frac{3}{\log_x 2} = 6$

$\Rightarrow \frac{1}{\log_x 2}(1+3) = 6 \Rightarrow \frac{1}{\log_x 2}(4) = 6$

$\Rightarrow \frac{1}{\log_x 2} = \frac{6}{4} = \frac{3}{2}$

$\Rightarrow \log_2 x = \frac{3}{2} \Rightarrow 2^{\frac{3}{2}} = x$

(Exponential form)

$\Rightarrow (2^3)^{1/2} = x \Rightarrow (8)^{1/2} = x$

$\Rightarrow x = \sqrt{8} = \sqrt{4\times 2}$

$\Rightarrow x = 2\sqrt{2}$.

12. **Solve $\log_{5-x}(x^2 - 6x + 65) = 2$.** [Hy. - 2018]

Sol : Given $\log_{5-x}(x^2 - 6x + 65) = 2$.

$(5-x)^2 = x^2 - 6x + 65$

[Converting into exponential form]

$\Rightarrow 25 + x^2 - 10x = x^2 - 6x + 65$

$\Rightarrow -6x + 65 - 25 + 10x = 0$

$\Rightarrow 4x + 40 = 0$

$\Rightarrow 4x = -40 \Rightarrow \frac{-40}{4} = -10.$

$\therefore x = -10.$

Exercise 2.13

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER.

1. **If $|x + 2| \leq 9$, then x belongs to** [Govt. MQP - 2018; Aug. - 2022; QY-'24]

(1) $(-\infty,-7)$ (2) $[-11, 7]$

(3) $(-\infty,-7)\cup[11,\infty)$ (4) $(-11,7)$

[Ans : (2) $[-11, 7]$]

Hint : $|x + 2| \leq 9 \Rightarrow -9 \leq x + 2 \leq 9; -11 \leq x \leq 7;$ $x\in[-11,7]$

2. **Given that x, y and b are real numbers $x < y$, $b > 0$, then**

(1) $xb < yb$ (2) $xb > yb$

(3) $xb \leq yb$ (4) $\frac{x}{b}\geq\frac{y}{b}$

Hint : $x < y, b > 0 \Rightarrow xb < yb$ [Ans : (1) $xb < yb$]

3. **If $\frac{|x-2|}{x-2} \geq 0$, then x belongs to**

(1) $[2,\infty)$ (2) $(2,\infty)$

(3) $(-\infty, 2)$ (4) $(-2,\infty)$

Hint : In $(2,\infty)$, $\frac{|x-2|}{x-2}$ is positive [Ans : (2) $(2,\infty)$]

4. **The solution of $5x - 1 < 24$ and $5x + 1 > -24$ is** [Sep. - 2020; First Mid - 2018; CRT - 2022; QY-'23 & '24]

(1) $(4, 5)$ (2) $(-5,-4)$

(3) $(-5, 5)$ (4) $(-5, 4)$

[Ans : (3) $(-5,5)$]

Hint :

$5x - 1 < 24$	$5x + 1 > -24$
$5x < 25$	$5x > -25$
$x < 5$	$x > -5$

$-5 < x < 5$

5. **The solution set of the following inequality $|x - 1| \geq |x - 3|$ is** [Qy. - 2018 & 2019]

(1) $[0, 2]$ (2) $[2,\infty)$

(3) $(0, 2)$ (4) $(-\infty,2)$ [Ans : (2) $[2,\infty)$]

Hint : When $x = 0$, $|-1| \geq |-3|$ not true

$\therefore$ (1) cannot be correct

When $x = 1$, $|0| \geq |-2|$ not true

$\therefore$ (3) cannot be correct

When $x = -4$, $|-5| \geq |-7|$ not true

$\therefore$ (4) cannot be correct

When $x = 3$, $|2| \geq |0|$ is true. $\therefore$ (2) is correct

6. The value of $\log_{\sqrt{2}} 512$ is **[May - 2022]**

(1) 16 (2) 18 (3) 9 (4) 12

Hint : Let $\log_{\sqrt{2}} 512 = x$ **[Ans : (2) 18]**

Then $\left(\sqrt{2}\right)^x = 512$

$\left(\sqrt{2}\right)^x = 2^9 \Rightarrow 2^{\frac{x}{2}} = 2^9 \Rightarrow \frac{x}{2} = 9$

$\Rightarrow x = 18$

7. The value of $\log_3 \frac{1}{81}$ is **[CRT - 2022]**

(1) –2 (2) –8 (3) –4 (4) –9

[Ans : (3) –4]

Hint : Let $\log_3 \frac{1}{81} = x \Rightarrow 3^x = \frac{1}{81} = 3^x = 3^{-4}$

$x = -4$

8. If $\log_{\sqrt{x}} 0.25 = 4$, then the value of x is

(1) 0.5 (2) 2.5 (3) 1.5 (4) 1.25

Hint : $\log_{\sqrt{x}} 0.25 = 4$ **[Ans : (1) 0.5]**

$\left(\sqrt{x}\right)^4 = 0.25$

$\left(\sqrt{x}\right)^4 = \frac{1}{4}$

$x^2 = \frac{1}{4} \Rightarrow x = \frac{1}{2} = 0.5$

9. The value of $\log_a b \ \log_b c \ \log_c a$ is **[CRT - 2022]**

(1) 2 (2) 1 (3) 3 (4) 4

[Ans : (2) 1]

Hint : $\log_a b \ \log_b c \ \log_c a = \frac{\log b}{\log a} \times \frac{\log c}{\log b} \times \frac{\log a}{\log c}$

$= 1 \quad \left[\because \log_b a = \frac{\log_c a}{\log_c b}\right]$

10. If 3 is the logarithm of 343, then the base is **[Sep. - 2021]**

(1) 5 (2) 7 (3) 6 (4) 9

Hint : $\log_x 343 = 3 \Rightarrow x^3 = 343 = 7^3$ **[Ans : (2) 7]**

$x = 7$

11. Find a so that the sum and product of the roots of the equation $2x^2 + (a-3)x + 3a - 5 = 0$ are equal is

(1) 1 (2) 2 (3) 0 (4) 4

Hint : $2x^2 + (a-3)x + 3a - 5 = 0$ **[Ans : (2) 2]**

Sum $= \frac{-(a-3)}{2} = \frac{-a+3}{2}$

Product $= \frac{3a-5}{2}$

Given they are equal, $\Rightarrow \frac{-a+3}{2} = \frac{3a-5}{2}$

$4a = 8$

$a = 2$

12. If a and b are the roots of the equation $x^2 - kx + 16 = 0$ and satisfy $a^2 + b^2 = 32$, then the value of k is
[Govt. MQP & First Mid - 2018; Hy. - 2019; Qy. - 2023]

(1) 10 (2) –8 (3) –8, 8 (4) 6

Hint : $a + b = k \,;\, ab = 16$ **[Ans : (3) –8, 8]**

$a^2 + b^2 = (a+b)^2 - 2ab$

$32 = k^2 - 32$

$k^2 = 64$

$k = \pm 8 \Rightarrow k = -8, 8$

13. The number of solutions of $x^2 + |x-1| = 1$ is
[Sep. - 2020; June - 2019; July-'24]

(1) 1 (2) 0 (3) 2 (4) 3

Hint : $x^2 + |x-1| = 1$ **[Ans : (3) 2]**

$|x-1| = 1 - x^2$

$\therefore 1 - x^2 = x - 1$	$1 - x^2 = -x + 1$
$x^2 + x - 2 = 0$	$x^2 - x = 0$
$(x+2)(x-1) = 0$	$x(x-1) = 0$
$x = -2$ (or) 1	$x = 0; \quad x = 1$

$\therefore$ The roots are 0, 1, –2

When $x = -2$, $x^2 + |x-1| = (-2)^2 + |-2-1|$

$= 4 + |-3| = 7 \neq 1$

Since 2 does not satisfy, the roots are 0 and 1

14. The equation whose roots are numerically equal but opposite in sign to the roots of $3x^2 - 5x - 7 = 0$ is

(1) $3x^2 - 5x - 7 = 0$ (2) $3x^2 + 5x - 7 = 0$
(3) $3x^2 - 5x + 7 = 0$ (4) $3x^2 + x - 7$

[Ans : (2) $3x^2 + 5x - 7 = 0$]

Hint : $3x^2 - 5x - 7 = 0$

Let the roots be $\alpha, \beta \Rightarrow$ Sum : $\alpha + \beta = \frac{5}{3}$

Product : $\alpha\beta = -\frac{7}{3}$

Now take the roots are $-\alpha, -\beta$

Sum : $-\alpha - \beta = -\frac{5}{3}$

Product : $(-\alpha)(-\beta) = -\frac{7}{3}$

The required equation $x^2 + \frac{5}{3}x - \frac{7}{3} = 0$

$3x^2 + 5x - 7 = 0$

15. If 8 and 2 are the roots of $x^2 + ax + c = 0$ and 3, 3 are the roots of $x^2 + dx + b = 0$; then the roots of the equation $x^2 + ax + b = 0$ are

(1) 1, 2 (2) –1, 1 (3) 9, 1 (4) –1, 2

[Ans : (3) 9,1**]**

Hint :

$x^2 + ax + c = 0$	$x^2 + dx + b = 0$
8 and 2 are the roots	3, 3 are the roots
$\therefore a = -10, c = 16$	$d = -6, b = 9$

$$x^2 + ax + b = 0$$
$$x^2 - 10x + 9 = 0 \Rightarrow (x-1)(x-9) = 0$$
$$x = 1 \text{ (or) } 9$$

16. If a and b are the real roots of the equation $x^2 - kx + c = 0$, then the distance between the points $(a, 0)$ and $(b, 0)$ is

(1) $\sqrt{k^2 - 4c}$ (2) $\sqrt{4k^2 - c}$

(3) $\sqrt{4c - k^2}$ (4) $\sqrt{k - 8c}$

[Ans : (1) $\sqrt{k^2 - 4c}$ **]**

Hint : $x^2 - kx + c = 0$; a and b are the roots

$$a + b = k, \; ab = c$$

To find distance $= \sqrt{(a-b)^2 + 0^2} = (a - b)$

$$= \sqrt{(a+b)^2 - 4ab} = \sqrt{k^2 - 4c}$$

17. If $\dfrac{kx}{(x+2)(x-1)} = \dfrac{2}{x+2} + \dfrac{1}{x-1}$, then the value of k is **[Hy. - 2019]**

(1) 1 (2) 2 (3) 3 (4) 4

[Ans : (3) 3**]**

Hint :

$$\frac{2}{x+2} + \frac{1}{x-1} = \frac{2x - 2 + x + 2}{(x+2)(x-1)}$$
$$\frac{kx}{(x+2)(x-1)} = \frac{3x}{(x+2)(x-1)}$$
$$k = 3$$

18. If $\dfrac{1-2x}{3+2x-x^2} = \dfrac{A}{3-x} + \dfrac{B}{x+1}$, then the value of A + B is **[Qy. - 2018 & '24]**

(1) $\dfrac{-1}{2}$ (2) $\dfrac{-2}{3}$ (3) $\dfrac{1}{2}$ (4) $\dfrac{2}{3}$

[Ans : (1) $\dfrac{-1}{2}$ **]**

Hint :

$$\frac{1-2x}{3+2x-x^2} = \frac{A}{3-x} + \frac{B}{x+1}$$
$$1 - 2x = A(x+1) + B(3-x)$$

Put $x = 3$, $\quad -5 = A(4) \Rightarrow A = -\dfrac{5}{4}$

Put $x = -1$, $\quad 3 = B(4) \Rightarrow B = \dfrac{3}{4}$

$$A + B = -\frac{5}{4} + \frac{3}{4} = \frac{-2}{4} = \frac{-1}{2}$$

19. The number of roots of $(x+3)^4 + (x+5)^4 = 16$ is **[July-2023]**

(1) 4 (2) 2 (3) 3 (4) 0

Hint : The degree of $(x+3)^4 + (x+5)^4 = 16$ is 4. Hence number of root is 4. **[Ans :** (1) 4**]**

20. The value of $\log_3 11 . \log_{11} 13 . \log_{13} 15 . \log_{15} 27 . \log_{27} 81$ is **[Hy.- 2018; Qy. -2023]**

(1) 1 (2) 2 (3) 3 (4) 4

[Ans : (4) 4**]**

Hint :

$$\frac{\log 11}{\log 3} \times \frac{\log 13}{\log 11} \times \frac{\log 15}{\log 13} \times \frac{\log 27}{\log 15} \times \frac{\log 81}{\log 27}$$

$$\left[\because \log_b a = \frac{\log_c a}{\log_c b}\right]$$

$$= \frac{\log 81}{\log 3} = \log_3 81 \Rightarrow \log_3 3^4 = 4 \log_3 3 = 4(1) = 4$$

$[\because \log_3 3 = 1]$

GOVERNMENT EXAM QUESTIONS

SECTION - A (1 MARK)

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER.

1. If quadratic with real coefficients has no real roots, then its discriminant is ________. **[Govt. MQP - 2018]**

(1) 0 (2) < 0 (3) >0 (4) 1

Hint : $b^2 - 4ac < 0$ **[Ans :** (2) < 0**]**

2. If $\sqrt{x+14} < 2$, then x belongs to **[Govt. MQP - 2018]**

(1) [–14, –10) (2) (–14, –10)

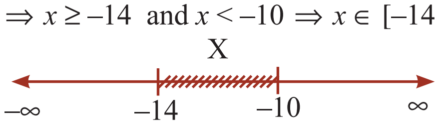
(3) (–∞, –10) (4) [–14, –10]

[Ans : (1) [–14, –10)**]**

Hint : $x + 14 \geq 0$ and $x + 14 < 4$

[Since squaring $\sqrt{x+14} < 2$ on both sides]

$\Rightarrow x \geq -14$ and $x < -10 \Rightarrow x \in [-14, -10)$



3. **For $x \geq 2$, $|x-2| =$** [First Mid - 2018]

(1) $2-x$ (2) $2+x$ (3) $x-2$ (4) x

[Ans : (3) $x-2$]

4. **The value of log 1 is** [First Mid - 2018]

(1) 1 (2) 0 (3) ∞ (4) -1

Hint : The value of log 1 is zero. [Ans : (2) 0]

5. **The value of $\sqrt[4]{(-2)^4} =$ ________.** [Qy. - 2018]

(1) 2 (2) -2 (3) 4 (4) -4

Hint : $\sqrt[4]{(-2)^4} = \sqrt[4]{16} = \sqrt[4]{2^4} = 2$ [Ans : (1) 2]

6. **If $|x-2| > 5$, then x belongs to** [Qy. - 2018]

(1) $(-\infty, -2] \cup [5, \infty)$ (2) $(-\infty, -3] \cup [7, \infty)$

(3) $(-\infty, -3) \cup (7, \infty)$ (4) $(-\infty, -2] \cup (5, \infty)$

[Ans : (3) $(-\infty, -3) \cup (7, \infty)$]

Hint : $x-2 > 5$ (or) $-(x-2) > 5$

$x > 7$ (or) $x - 2 < -5 \Rightarrow x < -5 + 2 \Rightarrow \Rightarrow x < -3$

7. **If $\left(\frac{2}{3}\right)^{x+2} = \left(\frac{3}{2}\right)^{2-2x}$ then $x =$** [Qy. - 2019]

(1) 1 (2) 3 (3) 4 (4) 0

[Ans : (3) 4]

Hint : $\left(\frac{2}{3}\right)^{x+2} = \left(\frac{3}{2}\right)^{2-2x} \Rightarrow \left(\frac{2}{3}\right)^{x+2} = \left(\frac{2}{3}\right)^{2x-2}$

$x + 2 = 2x - 2 \Rightarrow x = 4$

8. **The value of $\dfrac{2(3^{n+1}) + 7(3^{n-1})}{3^{n+2} - 2\left(\frac{1}{3}\right)^{1-n}}$** [Qy. - 2019]

(1) 1 (2) 3 (3) -1 (4) 0

[Ans : (1) 1]

Hint := $\dfrac{(2\times 3^n \times 3) + \left(7 \times 3^n \times \frac{1}{3}\right)}{(3^n \times 9) - \left(2 \times 3^n \times \frac{1}{3}\right)} = \dfrac{3^n\left(6+\frac{7}{3}\right)}{3^n\left(9-\frac{2}{3}\right)} = \dfrac{\frac{25}{3}}{\frac{25}{3}} = 1$

9. **If $|x+2| \leq 8$, then x belongs to :** [Mar. - 2019]

(1) (6, 10) (2) (– 10, 6)

(3) [6, 10] (4) [– 10, 6]

Hint : $-8 \leq x + 2 \leq 8$ [Ans : (4) [–10, 6]]

$-8 - 2 \leq x \leq 8 - 2$ $-10 \leq x \leq 6$

10. **If one root of the equation $x^2 + k = 0$ is negative of the other then find k.** [CRT - 2022]

(1) (2, –3) (2) (3, –3)

(3) (3, 3) (4) (–4, 3)

Hint : Consider $x^2 - 9 = 0$ [Ans : (*) $k = -9$]

$\Rightarrow$ The roots are – 3 and 3. So k is –9

11. **If the difference between the roots of the equation $x^2 + px + 8 = 0$ is 2 then the values of p is:** [July-2023]

(1) ± 4 (2) ± 5 (3) ± 6 (4) ± 7

[Ans : (3) ± 6]

12. **The logarithmic form of $5^2 = 25$ is** [Qy. - 2023]

(1) $\log_5 2 = 25$ (2) $\log_2 5 = 25$

(3) $\log_5 25 = 2$ (4) $\log_{25} 5 = 2$

[Ans : (3) $\log_5 25 = 2$]

SECTION - B (2 MARKS)

1. **Solve : $(x-2)(x+3)^2 < 0$.** [Govt. MQP - 2018]

Sol : $(x-2)(x+3)^2 < 0$ $\therefore$ Critical numbers are 2, – 3

We have three intervals $(-\infty, -3)$, $(-3, 2)$, $(2, \infty)$

Interval	Sign of $(x-2)$	Sign of $(x+3)^2$	Sign of $(x-2)(x+3)^2$
$(-\infty, -3)$ $x = -5$	–	+	–
$(-3, 2)$ $x = 0$	–	+	–
$(2, \infty)$ $x = 3$	+	+	+

The inequality is satisfied in the interval $(-\infty, -3)$ and $(-3, 2)$

$\therefore$ Solution set is $(-\infty, -3) \cup (-3, 2)$

2. **Prove that $\sum_{k=1}^{n} \dfrac{1}{k(k+1)} = 1 - \dfrac{1}{n+1}$.** [Govt. MQP - 2018]

Sol : Let t_k denote the k^{th} term of the given series.

Then $t_k = \dfrac{1}{k(k+1)}$. By using partial fraction we get

$\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ [Here A = 1; B = –1]

Thus $t_1 + t_2 + ... + t_n =$

$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \dfrac{1}{n+1}$

3. **Prove that $\log_4 2 - \log_8 2 + \log_6 2 - ...$ is $1 - \log_e 2$.** [Govt. MQP - 2018]

Sol : LHS $= \log_4 2 - \log_8 2 + \log_{16} 2 - \log_{32} 2 + ...$

$= \dfrac{1}{\log_2 4} - \dfrac{1}{\log_2 8} + \dfrac{1}{\log_2 16} - \dfrac{1}{\log_2 32} + ...$

$= \dfrac{1}{\log_2 2^2} - \dfrac{1}{\log_2 2^3} + \dfrac{1}{\log_2 2^4} - \dfrac{1}{\log_2 2^5} + ...$

$= \dfrac{1}{2\log_2 2} - \dfrac{1}{3\log_2 2} + \dfrac{1}{4\log_2 2} - \dfrac{1}{5\log_2 2} + ...$

[$\because \log_2 2 = 1$]

$$= \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+... = 1-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+...$$

[Add and subtract 1]

$$= 1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...\right) = 1-\log_e 2 = \text{RHS}.$$

4. Resolve into partial function $\frac{2}{x^2-1}$. **[First Mid - 2018]**

Sol :
$$\frac{2}{x^2-1} = \frac{2}{(x+1)(x-1)} = \frac{A}{x+1}+\frac{B}{x-1}$$
$$\frac{2}{x^2-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$
$$2 = A(x-1)+B(x+1) \quad ...(1)$$

putting $x = 1$ we get, $2 = B(1+1)$

$\Rightarrow B(2) = 2 \Rightarrow B = 1$

putting $x = -1$ in (1) we get $2 = A(-1-1)+B(-1+1)$

$2 = A(-2) \Rightarrow A = -1 \qquad \therefore \frac{2}{x^2-1} = -\frac{1}{x+1}+\frac{1}{x-1}$

5. Find the values of P for which the difference between the roots of the equation $x^2+px+8=0$ is 2. **[First Mid - 2018]**

Sol : Let α and β be the roots of the equation

$$x^2+px+8 = 0$$
Then, $\alpha+\beta = -p$, $\alpha\beta = 8$ and $|\alpha-\beta| = 2$

Now, $(\alpha+\beta)^2-4\alpha\beta = (\alpha-\beta)^2$ which gives

$$(-p)^2-4(8) = 2^2 \Rightarrow p^2-32 = 4$$
$$p^2 = 36 \Rightarrow p = \pm 6$$

6. Find the domain and range of the real valued function $f(x) = \frac{5-x}{x-5}$. **[Qy. - 2018]**

Sol : $f(x) \frac{5-2}{-(5-x)} = \frac{5-x}{-(5-x)} = -1$

Domain = R – {5}, Range = {–1}

7. Solve the equation $\sqrt{6-4x-x^2} = x+4$. **[Qy. - 2018; Mar. - 2019; April - 2023]**

Sol : The given equation is equivalent to the system $(x+4) \geq 0$ and $6-4x-x^2 = (x+4)^2$.

This implies $x \geq -4$ and $x^2+6x+5 = 0$.

On solving $x^2+6x+5 = 0$, we get $x = -1, -5$.

But only $x = -1$ satisfies both the conditions. Hence, $x = -1$.

8. If a and b are the roots of the equation $x^2-px+q=0$ find the value of $\frac{1}{a}+\frac{1}{b}$. **[Qy. - 2019; Sep. - 2021]**

Sol : Given that a and b are the roots of $x^2-px+q=0$. Then, $a+b = p$ and $ab = q$. Thus,

$$\frac{1}{a}+\frac{1}{b} = \frac{a+b}{ab} = \frac{p}{q}$$

9. Find the complete set of value of 'a' for which the quadratic $x^2-ax+a+2=0$ has equal roots. **[Sep. - 2020]**

Sol : The quadratic equation $x^2-ax+a+2=0$ has equal roots. So, its discriminant is zero.

Thus, $D = b^2-4ac = (-a)^2-4(a+2) = a^2-4a-8 = 0$

So, $a = \frac{4\pm\sqrt{48}}{2}$ which gives

$$a = \frac{4\pm4\sqrt{3}}{2} = \frac{\overset{2}{\cancel{4}}(1\pm1\sqrt{3})}{\underset{1}{\cancel{2}}} = 2(1\pm\sqrt{3})$$

10. Solve $|2x-17| = 3$ for x. **[May - 2022]**

Sol : $|2x-17| = 3$ Then we have $2x-17 = \pm 3$.

$2x-17 = -3$	$2x-17 = 3$
$2x = -3+17$	$2x = 3+17$
$2x = 14$	$2x = 20$
$x = 7$	$x = 10$

$x = 10$ or $x = 7$.

11. Solve $|x-9| < 2$ for x. **[Aug. - 2022]**

Sol : Refer Text Book Example 2.4

12. Rationalize the denominator of $\frac{\sqrt{5}}{\sqrt{6}+\sqrt{2}}$. **[Qy. - 2023]**

Sol : Refer Text Book Example 2.32

SECTION - C (3 MARKS)

1. If $\log_2 x+\log_4 x+\log_{16} x = \frac{7}{2}$ find the value of x. **[Qy. - 2019; July-2023]**

Sol : Note that $x > 0$.

$\log_2 x+\log_4 x+\log_{16} x = \frac{7}{2}$ becomes

$$\frac{1}{\log_x 2}+\frac{1}{\log_x 4}+\frac{1}{\log_x 16} = \frac{7}{2}. \text{ (change of base rule)}$$

Thus $\frac{1}{a}+\frac{1}{2a}+\frac{1}{4a} = \frac{7}{2}$ where $a = \log_x 2$.

That is $\frac{7}{4a} = \frac{7}{2} \Rightarrow a = \frac{1}{2}$

$$\therefore \frac{1}{2} = \log_x 2 \Rightarrow x^{\frac{1}{2}} = 2$$

$\Rightarrow$ $\boxed{x = 4}$

2. Find the value of $\frac{1}{\log_x(yz)+1}+\frac{1}{\log_y(zx)+1}+\frac{1}{\log_z(xy)+1}$ **[Qy-2019]**

Sol :
$$= \frac{1}{\log_x(yz)+1}+\frac{1}{\log_y(zx)+1}+\frac{1}{\log_z(xy)+1}$$
$$= \frac{1}{\frac{\log(yz)}{\log x}+1}+\frac{1}{\frac{\log(zx)}{\log y}+1}+\frac{1}{\frac{\log(xy)}{\log z}+1}$$

$$= \frac{\log x}{\log(yz)+\log x} + \frac{\log y}{\log(zx)+\log y} + \frac{\log z}{\log(xy)+\log z}$$

$$= \frac{\log x}{\log xyz} + \frac{\log y}{\log xyz} + \frac{\log z}{\log xyz}$$

$$= \frac{\log x + \log y + \log z}{\log xyz} = \frac{\log xyz}{\log xyz} = 1$$

3. Solve : $\frac{|x|-1}{|x|-3} \geq 0, x \in \mathbb{R}, x \neq \pm 3.$ **[Qy. - 2018]**

Sol : Critical points are –3, –1, 1, 3

Intervals are (–∞, –3), (–3, –1), (–1, 1), (1, 3), (3, ∞)

The equation $\frac{|x|-1}{|x|-3} \geq 0$ is satisfied at the interval

(–∞, –3)(–1, 1) (3, ∞)

∴ The solution set is (–∞, –3]∪ [–1, 1] ∪ [3, ∞)

4. Resolve into partial fractions : $\frac{x}{(x+3)(x-4)}$.

[Hy. - 2018; CRT & May - 2022]

Sol : Let $\frac{x}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}$

Where A and B constants.

Then, $\frac{x}{(x+3)(x-4)} = \frac{A(x-4)+B(x+3)}{(x+3)(x-4)}$,

which give x = A(x – 4)+ B(x + 3)

When x = 4, we have B = $\frac{4}{7}$

When x = –3, we have A = $\frac{3}{7}$

Hence, $\frac{x}{(x+3)(x-4)} = \frac{3}{7(x+3)} + \frac{4}{7(x-4)}$

5. Given that $\log_{10}2 = 0.30103$, $\log_{10}3 = 0.47712$ (approximately), find the number of digits in $2^8.3^{12}$.

[First Mid - 2018]

Sol : Suppose that N = $2^8 3^{12}$ has n + 1 digits.

Then N can be written as $10^n \times b$ where $1 \leq b < 10$.

Taking logarithm to the base 10, we get

$\log_{10} N = \log_{10}(10^n b)$

$= n\log_{10}10 + \log_{10} b = n + \log_{10} b$ $\left[\because \log_{10}^{10} = 1\right]$

On the other hand,

$\log_{10} N = \log_{10}2^8 3^{12} = 8\log_{10}2 + 12\log_{10}3$

$= (8 \times 0.30103) + (12 \times 0.47712)$

$= 8.13368$

Thus, we get $n + \log_{10} b = 8.13368$

Since $1 \leq b < 10$ the number of digit is 9.

6. Resolve into partial fractions : $\frac{10x+30}{(x^2-9)(x+7)}$.

[Govt. MQP - 2018]

Sol : Let $\frac{10x+30}{(x^2-9)(x+7)} = \frac{10(x+3)}{(x+3)(x-3)(x+7)}$

$= \frac{10}{(x-3)(x+7)} = \frac{A}{x-3} + \frac{B}{x+7} = \frac{A(x+7)+B(x-3)}{(x-3)(x+7)}$

∴ 10 = A (x + 7) + B (x – 3)

If x = 3 then A = 1

If x = –7 then B = –1

Hence, $\frac{10x+30}{(x^2-9)(x+7)} = \frac{1}{x-3} - \frac{1}{x+7}$

7. Solve : $x = \sqrt{x+20}$ for $x \in$ R **[QY-'23 & '24]**

Sol : Refer Text Book Example 2.21

SECTION - D (5 MARKS)

1. If x = 1 is one root of the equation $x^3 - 6x^2 + 11x - 6 = 0$, find the other roots. **[Qy. - 2018]**

Sol : Given, one of the root is x = 1

Using synthetic division we get,

1	1	–6	11	–6
	0	1	–5	6
	1	–5	6	0

The required equation is $(x-1)(x^2-5x+6) = 0$

⇒ $(x-1)(x-2)(x-3) = 0$ ⇒ x = 1, 2, 3

∴ The other roots are 2 and 3.

2. Prove that $\log\left(\frac{75}{16}\right) - 2\log\left(\frac{5}{9}\right) + \log\left(\frac{32}{243}\right) = \log 2.$

[May - 2022]

Sol : Using the properties of logarithm, we have

LHS = $\log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243}$

= log 75 – log 16 – 2 log 5 + 2 log 9 + log 32 – log 243 (Quotient rule)

= log3 + log25 – log16 – log25 + log81 + log16 + log 2 – log81 – log3

= log 2 = RHS

Hence proved.

3. Resolve into partial fractions $\frac{2x}{(x^2+1)(x-1)}$.

Sol : Refer Text Book Example 2.26 **[CRT - 2022]**

4. Resolve into partial fraction $\dfrac{1}{(x-1)(x^2-9)}$. [CRT - 2022]

Sol : $\dfrac{1}{(x-1)(x^2-9)} = \dfrac{1}{(x-1)(x^2-3^2)}$

$= \dfrac{1}{(x-1)(x+3)(x-3)}$

$\dfrac{1}{(x-1)(x+3)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+3)} + \dfrac{C}{(x-3)}$

$\dfrac{1}{(x-1)(x+3)(x-3)} = \dfrac{A(x+3)(x-3)+B(x-1)(x-3)+C(x-1)(x+3)}{(x-1)(x+3)(x-3)}$

Put $x = 3$, we get, $1 = C(2)(6) \Rightarrow \boxed{C = \frac{1}{12}}$

Put $x = 1$, we get, $1 = A(4)(-2) \Rightarrow \boxed{A = -\frac{1}{8}}$

Put $x = 0$, we get, $1 = A(3)(-3) + B(-1)(-3) + C(-1)(3)$

$1 = -9A + 3B - 3C$

$1 = -9\left(-\frac{1}{8}\right) + 3B - 3\left(\frac{1}{12}\right)$

$1 = \frac{9}{8} + 3B - \frac{1}{4}$

$1 - 3B = \frac{9-2}{8}$

$1 - 3B = \frac{7}{8}$

$-3B = \frac{7}{8} - 1$

$-3B = -\frac{1}{8}$

$B = \frac{1}{24}$

$\dfrac{1}{(x-1)(x^2-9)} = -\dfrac{1}{8(x-1)} + \dfrac{1}{24(x+3)} + \dfrac{1}{12(x-3)}$

5. Simplify : $\sqrt{x^2-10x+25}$ [Qy. - 2023]

Sol : Refer Text Book Example 2.31 (ii)

6. Resolve into partial fraction $\dfrac{x+1}{x^2(x-1)}$. [Qy. - 2023]

Sol : Refer Text Book Example 2.27

7. If $x = \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ and $y = \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ find the value of x^2+xy+y^2. [Hy. 2023]

Sol : $x = \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

$x = \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \times \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \dfrac{(\sqrt{3}-\sqrt{2})^2}{3-2} = \dfrac{3+2-2\sqrt{6}}{1}$

$x = 5-2\sqrt{6}$

$y = \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \times \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \dfrac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3})^2+(\sqrt{2})^2}$

$= \dfrac{3+2+2\sqrt{6}}{3-2} = 5+2\sqrt{6}$

$x^2+xy+y^2 = (5-2\sqrt{6})^2+(5-2\sqrt{6})(5+2\sqrt{6})+(5+2\sqrt{6})^2$

$= 25 + 24 - 20\sqrt{6} + 25 + 10\sqrt{6} - 10\sqrt{6} - 24 + 25 + 24 + 20\sqrt{6} = 99$

8. Resolve into partial fractions $\dfrac{2x}{(x^2+1)(x-1)}$ [Mar. - 2024]

Sol : Refer Text Book Example 2.26

ADDITIONAL PROBLEMS

SECTION - A (1 MARK)

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER.

1. Find the odd one out of the following

(1) $x^3 + 3x^2 + 2x + 1$ (2) $(x^2 + 2x + 1)(x + 4)$

(3) $x^2 + 5x + 6$ (4) $(x + 2)(x + 3)(x + 4)$

Hint : (3) is of degree 2 and all others are degree 3

[**Ans :** (3) $x^2 + 5x + 6$]

2. Assertion (A) : A plumber will be paid according to following schemes : In first scheme he will be paid ₹500 plus ₹70 her hour and in the second scheme he will paid ₹120 per hour. If he work for less than 10 hours the first scheme give better wages.

Reason (R) : In the first scheme he will get a fixed ₹500 whereas longer than 10 hours second scheme will pay more.

(1) Both A and R are true and R is the correct explanation of A

(2) Both A and R are true and R is not a correct explanation of A

(3) A is true but R is false

(4) A is false but R is true

[**Ans :** (1) Both A and R are true and R is the correct explanation of A]

SECTION - C (3 MARKS)

1. Solve the linear in equation $4 - x \leq 3x + 12$.

Sol : Given $4 - x \leq 3x + 12$

$\Rightarrow -x \leq 3x + 12 - 4$ $\quad \Rightarrow -x \leq 3x + 8$

$\Rightarrow 3x + 8 \geq -x$ $\quad [x \leq y \Rightarrow y \geq x]$

$\Rightarrow 4x + 8 \geq 0 \Rightarrow 4x \geq -8 \Rightarrow x \geq \frac{-8}{4} \quad \Rightarrow x \geq -2$

$\therefore x \in [-2, \infty)$ $\quad \therefore$ The solution set is $[-2, \infty)$.

SECTION - D (5 MARKS)

1. Solve: $\frac{2x+5}{x-1} > 5$

Sol : Given inequality is $\frac{2x+5}{x-1} > 5 \Rightarrow \frac{2x+5}{x-1} - 5 > 0$

$\Rightarrow \frac{2x+5-5x+5}{x-1} > 0 \Rightarrow \frac{-3x+10}{x-1} > 0$

$-3x + 10 > 0, x - 1 > 0 \Rightarrow -3x > -10, x > 1$

$3x < 10, x > 1 \Rightarrow x < \frac{10}{3}, x > 1$

$1 < x < \frac{10}{3}$ $\quad \therefore$ The solution is $\left(1, \frac{10}{3}\right)$

2. Solve: $\sqrt{x+5} + \sqrt{x+21} = \sqrt{6x+40}$

Sol : $\left(\sqrt{x+5} + \sqrt{x+21}\right)^2 = \left(\sqrt{6x+40}\right)^2$ (Squaring)

$\Rightarrow x + 5 + x + 21 + 2\sqrt{(x+5)(x+21)} = 6x + 40$

$\Rightarrow 2\sqrt{(x+5)(x+21)} = 4x + 14$

$\Rightarrow \sqrt{(x+5)(x+21)} = 2x + 7$

$(x + 5)(x + 21) = (2x + 7)^2$ (Again Squaring)

$\Rightarrow x^2 + 21x + 5x + 105 = 4x^2 + 49 + 28x$

$\Rightarrow 3x^2 + 2x - 56 = 0$

$\Rightarrow x = 4, \frac{-14}{3}$ (By factorize)

Case (i) When $x = 4$,

$\sqrt{4+5} + \sqrt{4+21} = \sqrt{6(4)+40}$

$\sqrt{9} + \sqrt{25} = \sqrt{64} \Rightarrow 3 + 5 = 8 \Rightarrow 8 = 8$

$\Rightarrow x = 4$ is a root.

Case (ii) When $x = \frac{-14}{3}$

$\sqrt{\frac{-14}{3} + 5} + \sqrt{\frac{-14}{3} + 21} = \sqrt{\overset{2}{\cancel{6}}\left(\frac{-14}{\cancel{3}}\right) + 40}$

$\sqrt{\frac{1}{3}} + \sqrt{\frac{49}{3}} = \sqrt{12}$

Which is not true. $\therefore x = \frac{-14}{3}$ is not a root.

$\therefore$ The only root is 4.

3. If $\frac{\log_e x}{b-c} = \frac{\log_e y}{c-a} = \frac{\log_e z}{a-b}$, show that
(i) $xyz = 1$ (ii) $x^a y^b z^c = 1$

Sol : Let $\frac{\log_e x}{b-c} = \frac{\log_e y}{c-a} = \frac{\log_e z}{a-b} = k$

$\log_e x = k(b - c)$, $\log_e y = k(c - a)$ and

$\log_e z = k(a - b)$ ...(1)

$\Rightarrow x = e^{k(b-c)}$,

$y = e^{k(c-a)}$ and $z = e^{k(a-b)}$ ...(2)

(i) $xyz = e^{k(b-c)} \cdot e^{k(c-a)} \cdot e^{k(a-b)}$

$= ek^{(\cancel{b} - \cancel{c} + \cancel{c} - \cancel{a} + \cancel{a} - \cancel{b})}$

$= e^{k(0)} = e^0 = 1 \Rightarrow xyz = 1$

(ii) $x^a y^b z^c = 1$

$x^a y^b z^c = [e^{k(b-c)}]^a \cdot [e^{k(c-a)}]^b \cdot [e^{k(a-b)}]^c$

$= e^{k(b-c).a} \cdot e^{k(c-a).b} \cdot e^{k(a-b).c}$

$= e^{k(\cancel{ab} - \cancel{ac} + \cancel{bc} - \cancel{ab} + \cancel{ac} - \cancel{bc})}$,

$= e^{k(0)} = e^0 = 1$

$\Rightarrow x^a y^b z^c = 1$

Hence proved.

❁❁❁❁❁