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It gives me great pride and pleasure in bringing to you **Sura's Business Mathematics Guide** for +1 Standard. A deep understanding of the text and exercises is rudimentary to have an insight into Business Mathematics. The students and teachers have to carefully understand the topics and exercises.

Sura's Business Mathematics +1 Standard Guide encompasses all the requirements of the students to comprehend the text and the evaluation of the textbook.

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I sincerely believe this guide satisfies the needs of the students and bolsters the teaching methodologies of the teachers.

I pray the almighty to bless the students for consummate success in their examinations.

Subash Raj, B.E., M.S.
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01 MATRICES AND DETERMINANTS

TEXTUAL QUESTIONS

EXERCISE 1.1

1. Find the minors and cofactors of all the elements of the following determinants.

(i) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$ [CRT - 2022] (ii) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Sol : (i) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$

$$\text{Let } A = \begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$$

Minor of 5 $= M_{11} = -1$

Minor of 20 $= M_{12} = 0$

Minor of 0 $= M_{21} = 20$

Minor of –1 $= M_{22} = 5$

Co-factor of 5 $= A_{11} = (-1)^{1+1} M_{11}$

$= (-1)^2 (-1) = -1$

Co-factor of 20 $= A_{12} (-1)^{1+2} M_{12} = -0 = 0$

Co-factor of 0 $= A_{21} (-1)^{2+1} M_{21} = 20$

Co-factor of –1 $= A_{22} (-1)^{2+2} M_{22} = 5$

(ii) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

$$\text{Let } B = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

Minor of 1 $= M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -2 - 10 = -12$

Minor of –3 $= M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 8 - 6 = 2$

Minor of 2 $= M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 20 + 3 = 23$

Minor of 4 $= M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -6 - 10 = -16$

Minor of –1 $= M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 6 = -4$

Minor of 2 $= M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 5 + 9 = 14$

Minor of 3 $= M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -6 + 2 = -4$

Minor of 5 $= M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 2 - 8 = -6$

Minor of 2 $= M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = -1 + 12 = 11$

Co-factor of 1 $= A_{11} = (-1)^{1+1} M_{11} = -12$

Co-factor of –3 $= A_{12} = (-1)^{1+2} M_{12} = 2$

Co-factor of 2 $= A_{13} = (-1)^{1+3} M_{13} = 23$

Co-factor of 4 $= A_{21} = (-1)^{2+1} M_{21} = -16$

Co-factor of –1 $= A_{22} = (-1)^{2+2} M_{22} = -4$

Co-factor of 2 $= A_{23} = (-1)^{2+3} M_{23} = 14$

Co-factor of 3 $= A_{31} = (-1)^{3+1} M_{31} = -4$

Co-factor of 5 $= A_{32} = (-1)^{3+2} M_{32} = -6$

Co-factor of 2 $= A_{33} = (-1)^{3+3} M_{33} = 11$

2. Evaluate : $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ **[Qy. - 2023]**

Sol : Let $A = \begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Expanding along R_1 we get,

$$|A| = 3\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$

$= 3(0-2) + 2(6-1) + 4(4-0)$

$= 3(-2) + 2(5) + 4(4)$

$= -6 + 10 + 16 = 20$

3. Solve : $\begin{vmatrix} 2 & x & 3 \\ 4 & 1 & 6 \\ 1 & 2 & 7 \end{vmatrix} = 0$ **[Sep. - 2021]**

Sol : Expanding along R_1 we get,

$$2\begin{vmatrix} 1 & 6 \\ 2 & 7 \end{vmatrix} - x\begin{vmatrix} 4 & 6 \\ 1 & 7 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow 2(7-12) - x(28-6) + 3(8-1) = 0$

$\Rightarrow 2(-5) - x(22) + 3(7) = 0$

$\Rightarrow -10 - 22x + 21 = 0$

$\Rightarrow -22x = -11$

$\Rightarrow x = \frac{11}{22} = \frac{1}{2}$

4. Find |AB| if $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$. **[Sep. - 2021]**

Sol : $AB = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 9-1 & 0+2 \\ 6+1 & 0-2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 7 & -2 \end{bmatrix}$

$\therefore |AB| = \begin{vmatrix} 8 & 2 \\ 7 & -2 \end{vmatrix} = -16 - 14 = -30$

$\therefore |AB| = -30$

5. Solve : $\begin{vmatrix} 7 & 4 & 11 \\ -3 & 5 & x \\ -x & 3 & 1 \end{vmatrix} = 0$

Sol : Expanding the given determinant along R_1 we get

$$7\begin{vmatrix} 5 & x \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} -3 & x \\ -x & 1 \end{vmatrix} + 11\begin{vmatrix} -3 & 5 \\ -x & 3 \end{vmatrix} = 0$$

$\Rightarrow 7(5-3x) - 4(-3+x^2) + 11(-9+5x) = 0$

$\Rightarrow 35 - 21x + 12 - 4x^2 - 99 + 55x = 0$

$\Rightarrow -4x^2 + 34x - 52 = 0$

Dividing by -2 we get,

$2x^2 - 17x + 26 = 0$

Using factorization, the factors are $2x^2 - 17x + 26 = 0$

$2x^2 - 4x - 13x + 26 = 0$

$2x(x-2) - 13(x-2) = 0$

$(x-2)(2x-13) = 0$

$\Rightarrow x = 2$ or $x = \frac{13}{2}$

The values of x are $2, \frac{13}{2}$

6. Evaluate : $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

Sol : Let $A = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

$$A = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \text{ [Using property (6)]... (1)}$$

Consider $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$ $\begin{matrix} R_1 \to R_1 \to R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$

Taking $(a-b)(b-c)$ common from R_1 and R_2,

$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$

$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$

$= (a-b)(b-c)[0 - 0 + \{b + c - (a+b)\}]$ [Expanding along C_1]

$= (a-b)(b-c)(c-a)$

Let $B = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Applying the elementary transformations.

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ we get,

$$B = \begin{vmatrix} 0 & a-b & bc-ca \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 0 & a-b & -c(a-b) \\ 0 & b-c & -a(b-c) \\ 1 & c & ab \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from R_1 and R_2 we get

$$B = (a-b)(b-c)\begin{vmatrix} 0 & 1 & -c \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding along C_1 we get,

$$B = (a-b)(b-c)(1)\begin{vmatrix} 1 & -c \\ 1 & -a \end{vmatrix}$$

$$= (a-b)(b-c)(-a+c)$$

$$= (a-b)(b-c)(c-a) \quad \ldots (2)$$

Substituting (2) in (1) we get,

$A = (a-b)(b-c)(c-a) - (a-b)(b-c)(c-a) = 0$

7. Prove that $\begin{vmatrix} \frac{1}{a} & bc & b+c \\ \frac{1}{b} & ca & c+a \\ \frac{1}{c} & ab & a+b \end{vmatrix} = 0.$ **[Aug. - 2022; Qy. - 2023]**

Sol : LHS = $\begin{vmatrix} \frac{1}{a} & bc & b+c \\ \frac{1}{b} & ca & c+a \\ \frac{1}{c} & ab & a+b \end{vmatrix}$

Multiplying R_1 by a, R_2 by b, and R_3 by c respectively and dividing the determinant by abc we get.

$$\text{LHS} = \frac{1}{abc}\begin{vmatrix} 1 & abc & ab+ac \\ 1 & abc & bc+ab \\ 1 & abc & ac+bc \end{vmatrix} = \frac{1}{abc}(abc)\begin{vmatrix} 1 & 1 & ab+ac \\ 1 & 1 & bc+ab \\ 1 & 1 & ac+bc \end{vmatrix}$$

[Taking abc common from C_2]

$= 0$ [$\because C_1 \equiv C_2$] = RHS.

Hence Proved.

8. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2 b^2 c^2$

[First Mid - 2018, Sep-2020]

Sol :

$$\text{LHS} = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Taking a, b, c common from R_1, R_2 and R_3 respectively we get.

$$\text{LHS} = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Again taking a, b, c common from C_1 C_2 and C_3 respectively.

$$\text{LHS} = a^2 b^2 c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying the elementary formation $R_1 \to R_1 + R_2$

$$= a^2 b^2 c^2\begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Now, expanding along R_1 we get

$$\text{LHS} = a^2 b^2 c^2 (2)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2a^2 b^2 c^2 (1+1)$$

$$= 4a^2 b^2 c^2 = \text{RHS}$$

Hence proved.

EXERCISE 1.2

1. Find the adjoint of the matrix A = $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Sol : Given A = $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

[Interchange the elements in the leading diagonal and change the sign in the off diagonal elements]

$$[A_{ij}] = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

2. If A = $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **then verify that A (*adj* A) = |A| I and also find A^{-1}.** **[GMQP - 2019]**

Sol : Given A = $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\text{Co-factor matrix } A_{ij} = \begin{bmatrix} +\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \\ +\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 16-9 & -(4-3) & +(3-4) \\ -(12-9) & +(4-3) & -(3-3) \\ +(9-12) & -(3-3) & +(4-3) \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Now, $\text{adj } A = [A_{ij}]^T$

$$= \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= 16 - 9 - 3(4-3) + 3(3-4)$$
$$= 7 - 3 - 3 = 7 - 6$$
$$= 1$$

Now, $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I$$

$[\because |A| = 1]$

Hence $A(\text{adj } A) = |A|.\, I$

Since $|A| \neq 0$, A^{-1} exists.

We know that

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

3. Find the inverse of each of the following matrices.

(i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ **(iv)** $\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$

Sol :

(i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements]

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5$$

Since $|A| \neq 0$, A^{-1} exists.

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

(ii) $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

Let $B = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

$$|B| = \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} = 9 + 1 = 10$$

Since $|B| \neq 0$, B^{-1} exists.

$$\text{adj } B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements]

$$\text{Now, } B^{-1} = \frac{1}{|B|}\text{adj } B = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

$$|C| = 1\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} + 3\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

[Expanding along R_1]

$$= 1(10-0) - 2(0-0) + 3(0-0)$$
$$= 10$$

Since $|C| \neq 0$, C^{-1} exists.

$$\text{adj } C = \begin{bmatrix} +\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 10-0 & -(0-0) & +(0-0) \\ -(10-0) & +(5-0) & -(0-0) \\ +(8-6) & -(4-0) & +(2-0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}^T = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Now, } C^{-1} = \frac{1}{|C|} \text{adj } C = \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

(iv) $\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$

$$\text{Let } D = \begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$$

$$|D| = \begin{vmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{vmatrix}$$

$$= -3\begin{vmatrix} 3 & -1 \\ -4 & -6 \end{vmatrix} + 5\begin{vmatrix} -2 & -1 \\ 1 & -6 \end{vmatrix} + 4\begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix}$$

[Expanding along R_1]

$$= -3(-18-4) + 5(12+1) + 4(8-3)$$
$$= -3(-22) + 5(13) + 4(5)$$
$$= 66 + 65 + 20 = 151$$

Since $|D| \neq 0$, D^{-1} exists.

$$\text{adj } D = \begin{bmatrix} +\begin{vmatrix} 3 & -1 \\ -4 & -6 \end{vmatrix} & -\begin{vmatrix} -2 & -1 \\ 1 & -6 \end{vmatrix} & +\begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix} \\ -\begin{vmatrix} -5 & 4 \\ -4 & -6 \end{vmatrix} & +\begin{vmatrix} -3 & 4 \\ 1 & -6 \end{vmatrix} & -\begin{vmatrix} -3 & -5 \\ 1 & -4 \end{vmatrix} \\ +\begin{vmatrix} -5 & 4 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} -3 & 4 \\ -2 & -1 \end{vmatrix} & +\begin{vmatrix} -3 & -5 \\ -2 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (-18-4) & -(12+1) & +(8-3) \\ -(30+16) & +(18-4) & -(12+5) \\ +(5-12) & -(3+8) & +(-9-10) \end{bmatrix}^T$$

$$= \begin{bmatrix} -22 & -13 & 5 \\ -46 & 14 & -17 \\ -7 & -11 & -19 \end{bmatrix}^T = \begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$$

$$\therefore D^{-1} = \frac{1}{|D|} \text{adj } D = \frac{1}{151} \begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$, then verify adj (AB) = (adj B) (adj A). **[CRT - 2022]**

Sol :

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 8-6 \\ -1-6 & 4+12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -7 & 16 \end{bmatrix}$$

Now, LHS

$$\text{adj } AB = \begin{bmatrix} 16 & -2 \\ 7 & 1 \end{bmatrix}$$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements] ... (1)

$$\text{Given } B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} -2 & -4 \\ -1 & -1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$$

$$\textit{adj } A = \begin{bmatrix} -6 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\text{RHS} = (\text{adj } B)(\text{adj } A)$$

$$= \begin{bmatrix} -2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+4 & 6-8 \\ 6+1 & 3-2 \end{bmatrix} = \begin{bmatrix} 16 & -2 \\ 7 & 1 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2)

$$\text{adj }(AB) = (\text{adj } B)(\text{adj } A)$$

5. If $A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$ then, show that (adj A)A = 0.

Sol : Given $A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 9 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 9 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 2 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ 9 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 9 & 1 \end{vmatrix} \\ +\begin{vmatrix} -2 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (15-0) & -(10-0) & +(2-27) \\ -(-10-2) & +(10-18) & -(2+18) \\ +(0-6) & -(0-4) & +(6+4) \end{bmatrix}^T$$

$$= \begin{bmatrix} 15 & -10 & -25 \\ 12 & -8 & -20 \\ -6 & 4 & 10 \end{bmatrix}^T = \begin{bmatrix} 15 & 12 & -6 \\ -10 & -8 & 4 \\ -25 & -20 & 10 \end{bmatrix}$$

Consider (adj A)A

$$= \begin{bmatrix} 15 & 12 & -6 \\ -10 & -8 & 4 \\ -25 & -20 & 10 \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 30+24-54 & -30+36-6 & 30+0-30 \\ -20-16+36 & 20-24+4 & -20+0+20 \\ -50-40+90 & 50-60+10 & -50+0+50 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence (adj A)A = 0

6. If $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$ then, show that the inverse of A is A itself. **[First Mid - 2018]**

Sol : Given $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{vmatrix}$$

$$= -1\begin{vmatrix} -3 & 4 \\ -4 & 5 \end{vmatrix} - 2\begin{vmatrix} 4 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 4 & -3 \\ 4 & -4 \end{vmatrix}$$

$$= -1(-15+16) - 2(20-16) - 2(-16+12)$$

$$= -1(1) - 2(4) - 2(-4)$$

$$= -1 - 8 + 8 = -1$$

Since $|A| \neq 0$, A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} -3 & 4 \\ -4 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 4 \\ 4 & 5 \end{vmatrix} & +\begin{vmatrix} 4 & -3 \\ 4 & -4 \end{vmatrix} \\ -\begin{vmatrix} 2 & -2 \\ -4 & 5 \end{vmatrix} & +\begin{vmatrix} -1 & -2 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 4 & -4 \end{vmatrix} \\ +\begin{vmatrix} 2 & -2 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ 4 & 4 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 4 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (-15+16) & -(20-16) & +(-16+12) \\ -(10-8) & +(-5+8) & -(4-8) \\ +(8-6) & -(-4+8) & +(3-8) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -4 & -4 \\ -2 & 3 & 4 \\ 2 & -4 & -5 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix} = A \Rightarrow A^{-1} = A \text{ itself}$$

7. If $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ then, find A.

Sol : Given $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

We know that $(A^{-1})^{-1} = A$

$\therefore$ We have to find $(A^{-1})^{-1}$

$$|A^{-1}| = 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1-1) + 3(-2-1) = -9$$

Since $|A^{-1}| \neq 0$, $(A^{-1})^{-1}$ exists.

$$\text{Now, adj } A^{-1} = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(1-1) & -(2+1) & +(-2-1) \\ -(0+3) & +(1-3) & -(-1-0) \\ +(0-3) & -(-1-6) & +(1-0) \end{bmatrix}^T = \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 1 \\ -3 & 7 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$(A^{-1})^{-1} = \frac{1}{|A^{-1}|}\text{adj}(A^{-1}) = \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A = \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix} \Rightarrow A = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

8. Show that the matrices A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$ are inverses of each other. **[Mar. - 2020 & 2023]**

Sol : Given A $= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{5} & \frac{-2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

Now AB $= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$

[Taking $\frac{1}{5}$ common from matrix B]

$= \frac{1}{5}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 8-2-1 & -4+6-2 & -2-2+4 \\ 4-3-1 & -2+9-2 & -1-3+4 \\ 4-2-2 & -2+6-4 & -1-2+8 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ... (1)

and BA $= \frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 8-2-1 & 8-6-2 & 4-2-2 \\ -2+3-1 & -2+9-2 & -1+3-2 \\ -2-2+4 & -2-6+8 & -1-2+8 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ... (2)

From (1) and (2), AB = BA = I

∴ The matrices A and B are inverses of each other.

9. If A = $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ then, verify that $(AB)^{-1} = B^{-1}A^{-1}$. **[Qy. - 2018 & 2023; Sep. - 2021]**

Sol : Given A $= \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and B $= \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix}$

$= 15 - 14 = 1 \neq 0 \Rightarrow A^{-1}$ exists.

$|B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = 54 - 56 = -2 \neq 0 \Rightarrow B^{-1}$ exists.

$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$

$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 4087 - 4089$

$= -2 \neq 0$ $(AB)^{-1}$ exists.

adj AB $= \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements]

$\Rightarrow (AB)^{-1} = \frac{1}{|AB|}$ adj (AB)

$= \frac{-1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & \frac{-67}{2} \end{bmatrix}$ ... (1)

$B^{-1} = \frac{1}{|B|}\text{adj } B = -\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{-1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$= \frac{-1}{2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = \begin{bmatrix} \frac{-61}{2} & \frac{87}{2} \\ \frac{47}{2} & \frac{-67}{2} \end{bmatrix}$ ... (2)

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$.

10. Find λ if the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{bmatrix}$ has no inverse.

Sol : Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{bmatrix}$ **[Qy. - 2018; Mar. - 2023]**

Since the matrix A has no-inverse, A is a singular matrix $\Rightarrow |A| = 0$.

$$|A| = \begin{vmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1\begin{vmatrix} \lambda & 4 \\ 7 & 11 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 9 & 11 \end{vmatrix} + 3\begin{vmatrix} 2 & \lambda \\ 9 & 7 \end{vmatrix} = 0$$

[Expanding along R_1]

$$11\lambda - 28 - 1(22 - 36) + 3(14 - 9\lambda) = 0$$
$$\Rightarrow \quad 11\lambda - 28 + 14 + 42 - 27\lambda = 0$$
$$\Rightarrow \quad -16\lambda + 28 = 0$$
$$\Rightarrow \quad 16\lambda = 28$$
$$\Rightarrow \quad \lambda = \frac{28}{16} = \frac{14}{8} = \frac{7}{4}$$
$$\Rightarrow \quad \therefore \lambda = \frac{7}{4}$$

11. If $X = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix}$ then, find p, q if $Y = X^{-1}$. **[Hy.-2019]**

Sol : Given $X = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix}$

Also it is given that $Y = X^{-1}$

$\Rightarrow \quad XY = XX^{-1}$ [Pre - multiply by X]

$\Rightarrow \quad XY = I$

$$\therefore \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16+0-15 & 8-2-3p & -8-1-3q \\ -10+0+10 & -5+2+2p & 5+1+2q \\ 20+0-20 & 10-2-4p & -10-1-4q \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 6-3p & -9-3q \\ 0 & -3+2p & 6+2q \\ 0 & 8-4p & -11-4q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Equating the corresponding entries on both sides, we get

$$6 - 3p = 0 \Rightarrow 6 = 3p \Rightarrow \frac{6}{3} = p$$
$$\therefore p = 2$$

Also, $-9 - 3q = 0 \Rightarrow -9 = 3q \Rightarrow \frac{-9}{3} = q$

$$\Rightarrow q = -3$$
$$\therefore p = 2 \text{ and } q = -3$$

Exercise 1.3

1. Solve by matrix inversion method : $2x + 3y - 5 = 0$, $x - 2y + 1 = 0$. **[First Mid - 2018; Qy. - 2023]**

Solution : Given equations are $2x + 3y = 5$, $x - 2y = -1$. Writing the given equations in matrix form we get,

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \quad \Rightarrow AX = B$$

$$\text{Where } A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B.$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$$
$$= -4 - 3 = -7 \neq 0 \Rightarrow A^{-1} \text{ exists.}$$

$$\text{adj } A = \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}$$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements]

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 10-3 \\ 5+2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad x = 1 \text{ and } y = 1$$

2. Solve by matrix inversion method :

(i) $3x - y + 2z = 13$; $2x + y - z = 3$; $x + 3y - 5z = -8$.

(ii) $x - y + 2z = 3$, $2x + z = 1$; $3x + 2y + z = 4$. **(GMQP-2019)**

(iii) $2x - z = 0$; $5x + y = 4$; $y + 3z = 5$. **[Qy.-2019]**

Sol : **(i) $3x - y + 2z = 13;\ 2x + y - z = 3;\ x + 3y - 5z = -8.$**

Given equations can be written in matrix form as

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \\ -8 \end{bmatrix} \Rightarrow AX = B$$

$$\text{Where } A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 13 \\ 3 \\ -8 \end{bmatrix} \Rightarrow X = A^{-1}B$$

$$\text{Now } |A| = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & -5 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

[Expanded along R_1]

$$= 3(-5+3) + 1(-10+1) + 2(6-1)$$
$$= 3(-2) + 1(-9) + 2(5)$$
$$= -5 \neq 0 \quad \Rightarrow A^{-1} \text{ exists.}$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -5 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-5+3) & -(-10+1) & +(6-1) \\ -(5-6) & +(-15-2) & -(9+1) \\ +(1-2) & -(-3-4) & +(3+2) \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 9 & 5 \\ 1 & -17 & -10 \\ -1 & 7 & 5 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{5}\begin{bmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \frac{-1}{5}\begin{bmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{bmatrix}\begin{bmatrix} 13 \\ 3 \\ -8 \end{bmatrix}$$

$$= \frac{-1}{5}\begin{bmatrix} -26+3+8 \\ 117-51-56 \\ 65-30-40 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -15 \\ 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore x = 3, y = -2, z = 1$

(ii) $x - y + 2z = 3;\ 2x + z = 1;\ 3x + 2y + z = 4.$

Given equations are $x - y + 2z = 3$, $2x + z = 1$ and $3x + 2y + z = 4$.

The given equations can be written in matrix form

$$\text{as } \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow AX = B \text{ Where } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 2\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= 1(0-2) + 1(2-3) + 2(4-0)$$
$$= -2 - 1 + 8 = 5 \neq 0 \Rightarrow A^{-1} \text{ exists.}$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0-2) & -(2-3) & +(4-0) \\ -(-1-4) & +(1-6) & -(2+3) \\ +(-1-0) & -(1-4) & +(0+2) \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 5 & -5 & -5 \\ -1 & 3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{5}\begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B$$

$$= \frac{1}{5}\begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -6+5-4 \\ 3-5+12 \\ 12-5+8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -5 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore x = -1, y = 2, z = 3$

(iii) $2x - z = 0, 5x + y = 4; y + 3z = 5.$

Given equations are $2x - z = 0$, $5x + y = 4$ and $y + 3z = 5$.

Given equations can be written in matrix form as

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad AX = B$

Where $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$

$\Rightarrow \quad X = A^{-1} B$

$$|A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2(3-0) - 0 - 1(5-0) = 6 - 5 = 1 \neq 0$$

$\Rightarrow A^{-1}$ exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(3-0) & -(15-0) & +(5-0) \\ -(0+1) & +(6-0) & -(2-0) \\ +(0+1) & -(0+5) & +(2-0) \end{bmatrix}^T = \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0-4+5 \\ 0+24-25 \\ 0-8+10 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \therefore x = 1, y = -1, z = 2$$

3. A sales person Ravi has the following record of sales for the month of January, February and March 2009 for three products A, B and C He has been paid a commission at fixed rate per unit but at varying rates for products A, B and C.

Months	Sales in Units			Commission
	A	B	C	
January	9	10	2	800
February	15	5	4	900
March	6	10	3	850

Find the rate of commission payable on A, B and C per unit sold using matrix inversion method.

Sol : Let x, y, z represent the rate of commission payable on A, B and C respectively.

Then $9x + 10y + 2z = 800$

$15x + 5y + 4z = 900$

$6x + 10y + 3z = 850$

The given equations can be written in matrix form as

$$\begin{bmatrix} 9 & 10 & 2 \\ 15 & 5 & 4 \\ 6 & 10 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$$

$\Rightarrow \quad AX = B$

Where $A = \begin{bmatrix} 9 & 10 & 2 \\ 15 & 5 & 4 \\ 6 & 10 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$

$$|A| = \begin{vmatrix} 9 & 10 & 2 \\ 15 & 5 & 4 \\ 6 & 10 & 3 \end{vmatrix} = 9\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} - 10\begin{vmatrix} 15 & 4 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 15 & 5 \\ 6 & 10 \end{vmatrix}$$

$= 9(15-40) - 10(45-24) + 2(150-30)$

$= 9(-25) - 10(21) + 2(120)$

$= -225 - 210 + 240 = -195 \neq 0 \Rightarrow A^{-1}$ exists.

$$\text{adj A} = \begin{bmatrix} +\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} & -\begin{vmatrix} 15 & 4 \\ 6 & 3 \end{vmatrix} & +\begin{vmatrix} 15 & 5 \\ 6 & 10 \end{vmatrix} \\ -\begin{vmatrix} 10 & 2 \\ 10 & 3 \end{vmatrix} & +\begin{vmatrix} 9 & 2 \\ 6 & 3 \end{vmatrix} & -\begin{vmatrix} 9 & 10 \\ 6 & 10 \end{vmatrix} \\ +\begin{vmatrix} 10 & 2 \\ 5 & 4 \end{vmatrix} & -\begin{vmatrix} 9 & 2 \\ 15 & 4 \end{vmatrix} & +\begin{vmatrix} 9 & 10 \\ 15 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(15-40) & -(45-24) & +(150-30) \\ -(30-20) & +(27-12) & -(90-60) \\ +(40-10) & -(36-30) & +(45-150) \end{bmatrix}^T$$

$$= \begin{bmatrix} -25 & -21 & 120 \\ -10 & 15 & -30 \\ 30 & -6 & -105 \end{bmatrix}^T = \begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj A} = \frac{-1}{195}\begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}$$

Also, $X = A^{-1} B$

$$= \frac{-1}{195}\begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}\begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$$

$$= \frac{-1}{195}\begin{bmatrix} -20,000-9,000+25,500 \\ -16,800+13,500-5,100 \\ 96,000-27,000-89,250 \end{bmatrix}$$

$$= \frac{-1}{195}\begin{bmatrix} -3500 \\ -8400 \\ -20,250 \end{bmatrix} = \begin{bmatrix} \frac{3500}{195} \\ \frac{8400}{195} \\ \frac{20,250}{195} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17.95 \\ 43.08 \\ 103.85 \end{bmatrix}$$

[Hence the rate of commission payable on A,B and C are 17.95, 43.08 and 103.85]

4. The prices of three commodities A, B and C are ₹ x, ₹ y and ₹ z per unit respectively. P purchases 4 units of C and sells 3 units of A and 5 units of B. Q purchases 3 units of B and sells 2 units of A and 1 unit of C. R purchases 1 unit of A and sells 4 units of B and 6 units of C. In the process P, Q and R earn ₹6,000 , ₹5,000 and ₹13,000 respectively. By using matrix inversion method, find the prices per unit of A, B and C.

Sol : By the given data

$$\begin{aligned} -4z + 3x + 5y &= 6000 \\ -3y + 2x + z &= 5000 \\ -x + 4y + 6z &= 13,000 \\ \Rightarrow 3x + 5y - 4z &= +6000 \\ 2x - 3y + z &= +5000 \\ -x + 4y + 6z &= 13,000 \end{aligned}$$

It can be written in matrix form as

$$\begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & +4 & +6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +6000 \\ +5000 \\ 13,000 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \text{ Where } A = \begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & +4 & +6 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} +6000 \\ +5000 \\ 13,000 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1} B$$

$$|A| = \begin{vmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & +4 & +6 \end{vmatrix}$$

$$= 3\begin{vmatrix} -3 & 1 \\ +4 & +6 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ -1 & +6 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -1 & +4 \end{vmatrix}$$

$= 3(-18-4) - 5(12+1) - 4(+8-3)$

$= 3(-22) - 5(13) - 4(5)$

$= -66 - 65 - 20 = -151 \neq 0 \Rightarrow A^{-1}$ exists

$$\text{adj A} = \begin{bmatrix} +\begin{vmatrix} -3 & 1 \\ +4 & +6 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -1 & +6 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ -1 & +4 \end{vmatrix} \\ -\begin{vmatrix} 5 & -4 \\ +4 & +6 \end{vmatrix} & +\begin{vmatrix} 3 & -4 \\ -1 & +6 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ -1 & +4 \end{vmatrix} \\ +\begin{vmatrix} 5 & -4 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-18-4) & -(+12+1) & +(+8-3) \\ -(+30+16) & +(+18-4) & -(+12+5) \\ +(5-12) & -(3+8) & +(-9-10) \end{bmatrix}^T$$

$$= \begin{bmatrix} -22 & -13 & +5 \\ -46 & +14 & -17 \\ -7 & -11 & -19 \end{bmatrix}^T = \begin{bmatrix} -22 & -46 & -7 \\ -13 & +14 & -11 \\ +5 & -17 & -19 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj A} = \frac{-1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & +14 & -11 \\ +5 & -17 & -19 \end{bmatrix}$$

Now, $X = A^{-1} B$

$$X = \frac{-1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}\begin{bmatrix} 6000 \\ 5000 \\ 13000 \end{bmatrix}$$

$$= \frac{-1}{151}\begin{bmatrix} -132000-230000-91000 \\ -78000+70000 \;\; 143000 \\ 30000-85000-247000 \end{bmatrix}$$

$$X = \frac{-1}{151}\begin{bmatrix} -453000 \\ -151000 \\ -302000 \end{bmatrix} = \begin{bmatrix} 3000 \\ 1000 \\ 2000 \end{bmatrix}$$

$\Rightarrow$ The prices of per unit of A, B, C are ₹ 3000, ₹ 1000 and ₹ 2000 respectively.

5. The sum of three numbers is 20. If we multiply the first by 2 and add the second number and subtract the third we get 23. If we multiply the first by 3 and add second and third to it, we get 46. By using matrix inversion method find the numbers. **[Hy. - 2019; Mar. - 2024]**

Sol : Let the required numbers be x, y and z.

$$\text{Given } x+y+z = 20$$
$$2x+y-z = 23$$
$$3x+y+z = 46$$

These equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix} \Rightarrow AX = B$$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$= 1\,(1+1) - 1\,(2+3) + 1\,(2-3)$

$= 2-5-1 = 2-6 = -4 \neq 0 \Rightarrow A^{-1}$ exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(1+1) & -(2+3) & +(2-3) \\ -(1-1) & +(1-3) & -(1-3) \\ +(-1-1) & -(-1-2) & +(1-2) \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -5 & -1 \\ 0 & -2 & 2 \\ -2 & 3 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -2 \\ -5 & -2 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & -2 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{-1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & -2 & 3 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} 40+0-92 \\ -100-46+138 \\ -20+46-46 \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -52 \\ -8 \\ -20 \end{bmatrix}$$

$X = \begin{bmatrix} 13 \\ 2 \\ 5 \end{bmatrix}$ $\therefore$ The required numbers are 13, 2 and 5.

6. Weekly expenditure in an office for three weeks is given as follows. Assuming that the salary in all the three weeks of different categories of staff did not vary, calculate the salary for each type of staff, using matrix inversion method.

Week	Number of employees			Total weekly Salary (in ₹)
	A	B	C	
1^{st} week	4	2	3	4900
2^{nd} week	3	3	2	4500
3^{rd} week	4	3	4	5800

Sol : Let the salary for each type of staff be x, y and z respectively.

Then, by the given data,

$$4x + 2y + 3z = 4900$$
$$3x + 3y + 2z = 4500$$
$$4x + 3y + 4z = 5800$$

These equations can be converted into matrix form as

$$\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4900 \\ 4500 \\ 5800 \end{bmatrix} \Rightarrow AX = B$$

Where $A = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4900 \\ 4500 \\ 5800 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix} = 4\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix}$$

$= 4\,(12-6) - 2\,(12-8) + 3\,(9-12)$

$= 4\,(6) - 2\,(4) + 3\,(-3) = 24-8-9 = 7 \neq 0$

$\therefore A^{-1}$ exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix}3&2\\4&4\end{vmatrix} & -\begin{vmatrix}3&2\\4&4\end{vmatrix} & +\begin{vmatrix}3&3\\4&3\end{vmatrix} \\ +\begin{vmatrix}2&3\\3&4\end{vmatrix} & +\begin{vmatrix}4&3\\4&4\end{vmatrix} & -\begin{vmatrix}4&2\\4&3\end{vmatrix} \\ +\begin{vmatrix}2&3\\3&2\end{vmatrix} & -\begin{vmatrix}4&3\\3&2\end{vmatrix} & +\begin{vmatrix}4&2\\3&3\end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(12-6)-(12-8)+(9-12) \\ -(8-9)+(16-12)-(12-8) \\ +(4-9)-(8-9)+(12-6) \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & -4 & -3 \\ 1 & 4 & -4 \\ -5 & 1 & 6 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -4 & 4 & 1 \\ -3 & -4 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -4 & 4 & 1 \\ -3 & -4 & 6 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -4 & 4 & 1 \\ -3 & -4 & 6 \end{bmatrix}\begin{bmatrix} 4900 \\ 4500 \\ 5800 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 29400+4500-29,000 \\ -19600+18000+5800 \\ -14700-18,000+34800 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4900 \\ 4200 \\ 2100 \end{bmatrix} = \begin{bmatrix} 700 \\ 600 \\ 300 \end{bmatrix}$$

Hence, the salary for each type of staff are ₹700, ₹600 and ₹300 respectively.

Exercise 1.4

1. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix}$. Test whether the system is viable as per Hawkins Simon conditions. **[Qy.-2018; Sep. - 2021]**

Sol : The technology matrix is $B = \begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix} = \begin{bmatrix} 0.50 & -0.30 \\ -0.41 & 0.67 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.50 & -0.30 \\ -0.41 & 0.67 \end{vmatrix}$$

$$= (0.50)(0.67) - (0.30)(0.41) = 0.335 - 0.123$$

$$= 0.212 > 0$$

Since the diagonals elements of I – B are positive and |I – B| is positive. Hawkins - Simon conditions are satisfied.

Hence the given system is viable.

2. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions. **[May - 2022]**

Sol : The technology matrix B is $\begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix}$

$$\therefore I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.9 \\ -0.20 & 0.20 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.4 & -0.9 \\ -0.20 & 0.20 \end{vmatrix} = (0.4)(0.20) - (0.20)(0.9)$$

$$= 0.08 - 0.18 = -0.1 < 0$$

Since |I – B| is negative, Hawkins – Simon conditions are not satisfied. Therefore the given system is not viable.

3. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions

Sol : The technology matrix is $B = \begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix} = \begin{bmatrix} 0.50 & -0.25 \\ -0.40 & 0.33 \end{bmatrix}$$

$$= (0.50)(0.33) - (0.40)(0.25)$$

$$= 0.165 - 0.1 = 0.065 > 0$$

Since the diagonal elements of (I – B) are positive and | I – B| is positive,

Hawkins – Simon conditions are satisfied.

Hence the given system is viable.

4. Two commodities A and B are produced such that 0.4 tonne of A and 0.7 tonne of B are required to produce a tonne of A. Similarly 0.1 tonne of A and 0.7 tonne of B are needed to produce a tonne of B. Write down the technology matrix. If 6.8 tonnes of A and 10.2 tonnes of B are required, find the gross production of both of them. **[Qy.-2019; GMQP-2018]**

Sol : The technology matrix B is $\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.7 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.3 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.6 & -0.1 \\ -0.7 & 0.3 \end{vmatrix} = (0.6)(0.3) - (0.7)(0.1)$$

$$= 0.18 - 0.07 = 0.11$$

$$\therefore (I-B)^{-1} = \frac{1}{|I-B|} \text{ adj } (I-B) = \frac{1}{0.11}\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

Since the diagonal elements of (I – B) are positive and |I – B| is positive, the system is viable.

Now $X = (I-B)^{-1} D$ where $D = \begin{bmatrix} 6.8 \\ 10.2 \end{bmatrix}$

$$X = \frac{1}{0.11}\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 6.8 \\ 10.2 \end{bmatrix}$$

$$= \frac{1}{0.11}\begin{bmatrix} 0.3\times 6.8 + 0.1\times 10.2 \\ 0.7\times 6.8 + 0.6\times 10.2 \end{bmatrix}$$

$$= \frac{1}{0.11}\begin{bmatrix} 2.04+1.02 \\ 4.76+6.12 \end{bmatrix}$$

$$= \frac{1}{0.11}\begin{bmatrix} 3.06 \\ 10.88 \end{bmatrix} = \begin{bmatrix} 27.82 \\ 98.91 \end{bmatrix}$$

∴ Gross production of commodity A and B are 27.82 and 98.91 tonnes.

5. Suppose the inter-industry flow of the product of two industries are given as under. **[Hy. - 2023]**

Production sector	Consumption sector		Domestic demand	Total output
	X	Y		
X	30	40	50	120
Y	20	10	30	60

Determine the technology matrix and test Hawkin's -Simon conditions for the viability of the system. If the domestic demand changes to 80 and 40 units respectively, what should be the gross output of each sector in order to meet the new demands.

Sol : $a_{11} = 30, a_{12} = 40, x_1 = 120$

$a_{21} = 20, a_{22} = 10, x_2 = 60$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{30}{120} = \frac{1}{4}$$

$$b_{12} = \frac{a_{12}}{x_2} = \frac{40}{60} = \frac{2}{3}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{20}{120} = \frac{1}{6}$$

$$b_{22} = \frac{a_{22}}{x_2} = \frac{10}{60} = \frac{1}{6}$$

The technology matrix is $B = \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

$$I-B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-2}{3} \\ \frac{-1}{6} & \frac{5}{6} \end{bmatrix}$$

$$|I-B| = \begin{vmatrix} \frac{3}{4} & \frac{-2}{3} \\ \frac{-1}{6} & \frac{5}{6} \end{vmatrix} = \left[\frac{3}{4}\right]\left[\frac{5}{6}\right] - \left[\frac{1}{6}\right]\left[\frac{2}{3}\right]$$

$$= \frac{15}{24} - \frac{2}{18} = \frac{5}{8} - \frac{1}{9} = \frac{45-8}{72} = \frac{37}{72} > 0$$

Main diagonals of I – B are positive and | I – B| is positive.

∴ The problem has a solution

$$(I-B)^{-1} = \frac{1}{|I-B|} \text{ adj } (I-B)$$

$$= \frac{1}{\frac{37}{72}}\begin{bmatrix} \frac{5}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{3}{4} \end{bmatrix} = \frac{72}{37}\begin{bmatrix} \frac{5}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{3}{4} \end{bmatrix}$$

$$X = (I-B)^{-1} D \text{ where } D = \begin{bmatrix} 80 \\ 40 \end{bmatrix}$$

$$= \frac{72}{37}\begin{bmatrix} \frac{5}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 80 \\ 40 \end{bmatrix} = \frac{72}{37}\begin{bmatrix} \frac{5}{6}\times 80 + \frac{2}{3}\times 40 \\ \frac{1}{6}\times 80 + \frac{3}{4}\times 40 \end{bmatrix}$$

$$= \frac{72}{37}\begin{bmatrix} \frac{400}{6} + \frac{80}{3} \\ \frac{80}{6} + \frac{120}{4} \end{bmatrix} = \frac{72}{37}\begin{bmatrix} \frac{400+160}{6} \\ \frac{320+720}{24} \end{bmatrix}$$

$$= \frac{72}{37}\begin{bmatrix} \frac{560}{6} \\ \frac{1040}{24} \end{bmatrix} = \begin{bmatrix} \frac{72}{37}\times\frac{560}{6} \\ \frac{72}{37}\times\frac{1040}{24} \end{bmatrix} = \begin{bmatrix} \frac{40320}{222} \\ \frac{74880}{888} \end{bmatrix}$$

$$= \begin{bmatrix} 181.62 \\ 84.32 \end{bmatrix}$$

The output for production section X and Y are 181.62 and 84.32 respectively.

6. You are given the following transaction matrix for a two sector economy. **[GMQP - 2019]**

Sector	Sales		Final demand	Gross output
	1	2		
1	4	3	13	20
2	5	4	3	12

i) Write the technology matrix.

ii) Determine the output when the final demand for the output sector 1 alone increases to 23 units.

Sol : $a_{11} = 4, a_{12} = 3, x_1 = 20$

$a_{21} = 5, a_{22} = 4, x_2 = 12$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{4}{20} = \frac{1}{5}; b_{12} = \frac{a_{12}}{x_2} = \frac{3}{12} = \frac{1}{4}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{5}{20} = \frac{1}{4}; b_{22} = \frac{a_{22}}{x_2} = \frac{4}{12} = \frac{1}{3}$$

i) The technology matrix is $B = \begin{bmatrix} \frac{1}{5} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{2}{3} \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} \frac{4}{5} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{2}{3} \end{vmatrix} = \left[\frac{4}{5}\right]\left[\frac{2}{3}\right] - \left[\frac{1}{4}\right]\left[\frac{1}{4}\right]$$

$$= \frac{8}{15} - \frac{1}{16} = \frac{128-15}{15\times16} = \frac{113}{240} > 0$$

(ii) Since the diagonals of (I – B) are positive and |I – B| is positive, the system has a solution.

$$\therefore (I - B)^{-1} = \frac{1}{|I - B|} \text{adj } (I - B)$$

$$= \frac{1}{\frac{113}{240}} \begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{4}{5} \end{bmatrix} = \frac{240}{113} \begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{4}{5} \end{bmatrix}$$

Also, $X = (I - B)^{-1} (D)$

Where $D = \begin{bmatrix} 23 \\ 3 \end{bmatrix} = \frac{240}{113} \begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 23 \\ 3 \end{bmatrix}$

$$= \frac{1}{113} \begin{bmatrix} \frac{2}{3}\times240 & \frac{1}{4}\times240 \\ \frac{1}{4}\times240 & \frac{4}{5}\times240 \end{bmatrix} \begin{bmatrix} 23 \\ 3 \end{bmatrix}$$

$$= \frac{1}{113} \begin{bmatrix} 160 & 60 \\ 60 & 192 \end{bmatrix} \begin{bmatrix} 23 \\ 3 \end{bmatrix}$$

$$= \frac{1}{113} \begin{bmatrix} 160\times23 + 60\times3 \\ 60\times23 + 192\times3 \end{bmatrix} = \frac{1}{113} \begin{bmatrix} 3680 + 180 \\ 1380 + 576 \end{bmatrix}$$

$$= \frac{1}{113} \begin{bmatrix} 3860 \\ 1956 \end{bmatrix} = \begin{bmatrix} 34.16 \\ 17.31 \end{bmatrix}$$

The output for sector 1 is 34.16 units and 23 is 17.31 units.

7. Suppose the inter-industry flow of the product of two sectors X and Y are given as under.

Production sector	Consumption Sector		Domestic demand	Gross output
	X	Y		
X	15	10	10	35
Y	20	30	15	65

Find the gross output when the domestic demand changes to 12 for X and 18 for Y. **[Qy. - 2018 & 2023; CRT - 2022]**

Sol : $a_{11} = 15, a_{12} = 10, x_1 = 35$

$a_{21} = 20, a_{22} = 30, x_2 = 65$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{15}{35} = \frac{3}{7}; b_{12} = \frac{a_{12}}{x_2} = \frac{10}{65} = \frac{2}{13}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{20}{35} = \frac{4}{7}; b_{22} = \frac{a_{22}}{x_2} = \frac{30}{65} = \frac{6}{13}$$

$\therefore$ The technology matrix is $B = \begin{bmatrix} \frac{3}{7} & \frac{2}{13} \\ \frac{4}{7} & \frac{6}{13} \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{7} & \frac{2}{13} \\ \frac{4}{7} & \frac{6}{13} \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{-2}{13} \\ \frac{-4}{7} & \frac{7}{13} \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} \frac{4}{7} & \frac{-2}{13} \\ \frac{-4}{7} & \frac{7}{13} \end{vmatrix} = \left[\frac{4}{7}\right]\left[\frac{7}{13}\right] - \left[\frac{4}{7}\right]\left[\frac{2}{13}\right]$$

$$= \frac{28}{91} - \frac{8}{91} = \frac{20}{91}$$

Since the diagonal elements of (I – B) are positive and |I – B| is positive, the system is viable

$$(I - B)^{-1} = \frac{1}{|I - B|} \text{adj } (I - B)$$

$$= \frac{1}{\frac{20}{91}}\begin{bmatrix}\frac{7}{13} & \frac{2}{13}\\ \frac{4}{7} & \frac{4}{7}\end{bmatrix} = \frac{91}{20}\begin{bmatrix}\frac{7}{13} & \frac{2}{13}\\ \frac{4}{7} & \frac{4}{7}\end{bmatrix}$$

Now, X $= (\mathrm{I}-\mathrm{B})^{-1}$ (D)

$$\text{where D} = \begin{bmatrix}12\\18\end{bmatrix} = \frac{91}{20}\begin{bmatrix}\frac{7}{13} & \frac{2}{13}\\ \frac{4}{7} & \frac{4}{7}\end{bmatrix}\begin{bmatrix}12\\18\end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix}49 & 14\\ 52 & 52\end{bmatrix}\begin{bmatrix}12\\18\end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix}49\times12+14\times18\\ 52\times12+52\times18\end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix}588+252\\ 624+936\end{bmatrix} = \frac{1}{20}\begin{bmatrix}840\\1560\end{bmatrix}$$

$$= \begin{bmatrix}42\\78\end{bmatrix}$$

$\therefore$ The gross output for two sectors X and Y are 42 and 78 respectively.

Exercise 1.5

Choose the correct answer:

1. The value of x if $\begin{vmatrix}0 & 1 & 0\\ x & 2 & x\\ 1 & 3 & x\end{vmatrix} = 0$ is **[May - 2022]**

(a) 0, –1 (b) 0, 1

(c) –1, 1 (d) –1, –1 **Ans: (b) 0, 1**

Hint:

$$\begin{aligned} -1[x^2-x] &= 0\\ 0 &= x^2-x\\ x(x-x) &= 0\\ x &= 0\,;x=1\end{aligned}$$

2. The value of $\begin{vmatrix}2x+y & x & y\\ 2y+z & y & z\\ 2z+x & z & x\end{vmatrix}$ is **[GMQP-2018, First Mid - 2018]**

(a) xyz (b) $x+y+z$

(c) $2x+2y+2z$ (d) 0 **Ans: (d) 0**

Hint: $= \begin{vmatrix}2x & x & y\\ 2y & t & z\\ 2z & z & x\end{vmatrix} + \begin{vmatrix}y & x & y\\ z & y & z\\ x & z & x\end{vmatrix} \begin{matrix}[\mathrm{C}_1 \equiv 2\mathrm{C}_2]\\ [\mathrm{C}_1 \equiv \mathrm{C}_3]\end{matrix}$

$= 0+0=0$

3. The co-factor of –7 in the determinant $\begin{vmatrix}2 & -3 & 5\\ 6 & 0 & 4\\ 1 & 5 & -7\end{vmatrix}$ is

(a) –18 (b) 18 **[May - 2022]**

(c) –7 (d) 7 **Ans: (b) 18**

Hint: $= (-1)^{3+3}\begin{vmatrix}2 & -3\\ 6 & 0\end{vmatrix}$

$= 2\times0-[(-3)\times6] = 0-(-18) = 18$

4. If $\Delta = \begin{vmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 3 & 1\end{vmatrix}$ then $\begin{vmatrix}3 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 1\end{vmatrix}$ is **[Mar. - 2020]**

(a) Δ (b) $-\Delta$

(c) 3Δ (d) -3Δ **Ans: (b) – Δ**

Hint: $\Delta = \begin{vmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 3 & 1\end{vmatrix} = -\begin{vmatrix}3 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 1\end{vmatrix} \mathrm{R}_1 \rightleftharpoons \mathrm{R}_2$

$= -\Delta$

5. The value of the determinant $\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}^2$ is

(a) abc (b) 0 **[CRT - 2022]**

(c) $a^2b^2c^2$ (d) $-abc$ **Ans: (c) $a^2b^2c^2$**

Hint: $= [a(bc-0)]^2 = a^2b^2c^2$

6. If A is a square matrix of order 3, then $|k\mathrm{A}|$ is

[First Mid - 2018, Sep. - 2020; CRT - 2022]

(a) $k|\mathrm{A}|$ (b) $-k|\mathrm{A}|$

(c) $k^3|\mathrm{A}|$ (d) $-k^3|\mathrm{A}|$ **Ans: (c) $k^3|\mathrm{A}|$**

Hint: $|k\mathrm{A}| = (k)\,(k)\,(k)\,|\mathrm{A}| = k^3|\mathrm{A}|$

7. adj (AB) is equal to **[June - 2019; Qy. - 2023]**

(a) adj A adj B (b) adj A^{T} adj B^{T}

(c) adj B adj A (d) adj B^{T} adj A^{T}

Ans: (c) adj B adj A

8. **The inverse matrix of** $\begin{pmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{pmatrix}$ **is**

[GMQP-2018; Qy.-2019; Aug. - 2022]

(a) $\frac{7}{30}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ (b) $\frac{7}{30}\begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix}$

(c) $\frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ (d) $\frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{4}{5} \end{pmatrix}$

Hint: $A = \begin{pmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{pmatrix}$ **Ans: (c)** $\frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$

$$|A| = \frac{2}{5}-\frac{1}{6} = \frac{12-6}{30} = \frac{7}{30}$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{2} \end{pmatrix}$$

9. **If** $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **such that** $ad - bc \neq 0$ **then** A^{-1} **is**

[Hy.-2019]

(a) $\frac{1}{ad-bc}\begin{pmatrix} d & b \\ -c & a \end{pmatrix}$ (b) $\frac{1}{ad-bc}\begin{pmatrix} d & b \\ c & a \end{pmatrix}$

(c) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (d) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ c & a \end{pmatrix}$

Hint: $|A| = \frac{1}{ad-bc}$; **Ans: (c)** $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

10. **The number of Hawkins-Simon conditions for the viability of an input – output analysis is**

[First Mid - 2018; Sep. - 2021; Mar. - 2023]

(a) 1 (b) 3

(c) 4 (d) 2 **Ans: (d) 2**

11. **The inventor of input–output analysis is**

[Hy.-2019; CRT & Aug. - 2022; Qy. - 2023; Mar. - 2024]

(a) Sir Francis Galton (b) Fisher

(c) Prof. Wassily W. Leontief

(d) Arthur Caylay

Ans: (c) Prof. Wassily W. Leontief

12. **Which of the following matrix has no inverse**

(a) $\begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}$ (d) $\begin{pmatrix} \sin a & \sin a \\ -\cos a & \cos a \end{pmatrix}$

Hint: $\begin{vmatrix} -2 & -1 \\ -4 & 2 \end{vmatrix} = 4-4=0$ **Ans: (b)** $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

13. **The Inverse matrix of** $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ **is** **[July - 2023]**

(a) $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ -5 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix}$

Ans: (a) $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

Hint: $|A| = 6-5 = 1$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

14. **If** $A = \begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}$ **then A (adj A) is** **[First Mid - 2018]**

(a) $\begin{pmatrix} -4 & -2 \\ -1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Ans: (c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Hint: $|A| = 4-2 = 2$

$$A(\text{adj } A) = |A|\, I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

15. If A and B non-singular matrix then, which of the following is incorrect? **[Qy.-2018]**

(a) $A^2 = I$ implies $A^{-1} = A$

(b) $I^{-1} = I$

(c) If AX = B, then $X = B^{-1} A$

(d) If A is square matrix of order 3 then $|\text{adj } A| = |A|^2$

Ans: (c) If AX = B, then $X = B^{-1} A$

Hint: $X = A^{-1}B \neq B^{-1}A$

16. The value of $\begin{vmatrix} 5 & 5 & 5 \\ 4x & 4y & 4z \\ -3x & -3y & -3z \end{vmatrix}$ **is**

(a) 5 (b) 4 (c) 0 (d) –3

Ans: (c) 0

Hint: $= (-5)(4)(-3)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x & y & z \end{vmatrix} \quad [C_2 \equiv C_3]$

17. If A is an invertible matrix of order 2 then det (A^{-1}) be equal to

(a) det (A) (b) $\dfrac{1}{\det(A)}$

(c) 1 (d) 0

Ans: (b) $\dfrac{1}{\det(A)}$

Hint:
$$AA^{-1} = I$$
$$|AA^{-1}| = 1$$
$$|A|\,|A^{-1}| = 1$$
$$|A^{-1}| = \frac{1}{|A|}$$

18. If A is 3 × 3 matrix and |A| = 4 then $|A^{-1}|$ is equal to

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{16}$ (c) 2 (d) 4

Hint: $|A^{-1}| = \dfrac{1}{|A|} = \dfrac{1}{4}$ **Ans: (a) $\dfrac{1}{4}$**

19. If A is a square matrix of order 3 and |A| = 3 then |adj A| is equal to **[GMQP - 2018; Qy. - 2023]**

(a) 81 (b) 27 (c) 3 (d) 9

Ans: (d) 9

Hint: Since $|\text{adj } A| = |A|^{n-1}\ [n = 3] = 3^2 = 9$

20. The value of $\begin{vmatrix} x & x^2 - yz & 1 \\ y & y^2 - zx & 1 \\ z & z^2 - xy & 1 \end{vmatrix}$ **is**

(a) 1 (b) 0 (c) –1 (d) $-xyz$

Ans: (b) 0

21. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$**, then |2A| is equal to** **[Qy.-2019; Sep. - 2021]**

(a) 4 cos 2θ (b) 4 **[First Mid - 2018]**

(c) 2 (d) 1

Ans: (b) 4

Hint: $= 2^2[\cos^2\theta + \sin^2\theta] = 4$

22. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ **and A_{ij} is cofactor of a_{ij}, then value of Δ is given by** **[Sep. - 2021]**

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Ans: (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Hint: Expanding along C1, we get

$a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

23. If $\begin{vmatrix} x & 2 \\ 8 & 5 \end{vmatrix} = 0$ **then the value of x is** **[Mar. - 2019 & 2024; July - 2023]**

(a) $\dfrac{-5}{6}$ (b) $\dfrac{5}{6}$ (c) $\dfrac{-16}{5}$ (d) $\dfrac{16}{5}$

Hint: $5x - 16 = 0 \Rightarrow x = \dfrac{16}{5}$ **Ans: (d) $\dfrac{16}{5}$**

24. If $\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = -5$ **then the value of** $\begin{vmatrix} 20 & 15 \\ 15 & 5 \end{vmatrix}$ **is** **[Qy.-2018]**

(a) –5 (b) –125 (c) –25 (d) 0

Hint: $5 \times 5\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = 5 \times 5 = (-5) = -125$

Ans: (b) –125

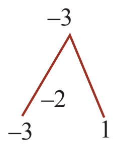
25. If any three rows or columns of a determinant are identical then the value of the determinant is **[Mar. - 2023]**

(a) 0 (b) 2 (c) 1 (d) 3

Hint: If any two rows of a determinant is identical then the value it is zero. **Ans: (a) 0**

Miscellaneous Problems

1. Solve : $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0.$



Sol : Expanding along R_1 we get

$$x\begin{vmatrix} 5 & x \\ 2 & x \end{vmatrix} - 2\begin{vmatrix} 2 & x \\ -1 & x \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} = 0$$

$$\begin{aligned}
\Rightarrow x(5x-2x) - 2(2x+x) - 1(4+5) &= 0 \\
\Rightarrow \quad x(3x) - 2(3x) - 1(9) &= 0 \\
\Rightarrow \quad 3x^2 - 6x - 9 &= 0 \\
\Rightarrow \quad x^2 - 2x - 3 &= 0 \quad \text{(Divided by 3)} \\
\Rightarrow \quad (x-3)(x+1) &= 0 \\
\Rightarrow \quad x &= 3 \text{ or } -1
\end{aligned}$$

2. Evaluate $\begin{vmatrix} 10041 & 10042 & 10043 \\ 10045 & 10046 & 10047 \\ 10049 & 10050 & 10051 \end{vmatrix}$

Sol : Applying the elementary transformation $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$ we get

$$\begin{vmatrix} 10041 & 10042 & 10043 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix} = 0 \qquad [R_2 \equiv R_3]$$

3. Without actual expansion show that the value of the determinant $\begin{vmatrix} 5 & 5^2 & 5^3 \\ 5^2 & 5^3 & 5^4 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ **is zero.**

Sol : Taking 5, 5^2 and 5^4 common from R_1, R_2 and R_3 respectively we get

$$\begin{vmatrix} 5 & 5^2 & 5^3 \\ 5^2 & 5^3 & 5^4 \\ 5^4 & 5^5 & 5^6 \end{vmatrix} = 5 \times 5^2 \times 5^4 \begin{vmatrix} 1 & 5 & 5^2 \\ 1 & 5 & 5^2 \\ 1 & 5 & 5^2 \end{vmatrix}$$

$$= 5 \times 5^2 \times 5^4 \times 0 = 0 \qquad [\because R_1 \equiv R_2 \equiv R_3]$$

4. Show that $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3 b^3 c^3.$

Sol :

$$\text{LHS} = \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$$

Taking a^2, b^2 and c^2 common from C_1, C_2 and C_3 respectively we get,

$$\text{LHS} = a^2 b^2 c^2 \begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix}$$

Again taking a, b, c common from R_1, R_2 and R_3 respectively we get,

$$\text{LHS} = a^3 b^3 c^3 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Expanding along R_1 we get

$$\begin{aligned}
\text{LHS} &= a^3b^3c^3\left[0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\right] \\
&= a^3b^3c^3\,[-1(0-1) + 1(1-0)] \\
&= a^3b^3c^3\,(1+1) \\
&= 2a^3b^3c^3 = \text{RHS.}
\end{aligned}$$

Hence proved.

5. If A = $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 7 \\ 1 & -1 & 1 \end{bmatrix}$ **verify that A (adj A) = (adj A) (A) = |A| I_3.**

Sol : Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 7 \\ 1 & -1 & 1 \end{bmatrix}$

$$\begin{aligned}
|A| &= 1\begin{vmatrix} 4 & 7 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 7 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} \\
&= (4+7) - 1(3-7) + 1(-3-4) \\
&= 11 + 4 - 7 = 8
\end{aligned}$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 4 & 7 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 7 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 4 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (4+7) & -(3-7) & +(-3-4) \\ -(1+1) & +(1-1) & -(-1-1) \\ +(7-4) & -(7-3) & +(4-3) \end{bmatrix}^T$$

$$= \begin{bmatrix} 11 & 4 & -7 \\ -2 & 0 & 2 \\ 3 & -4 & 1 \end{bmatrix}^T = \begin{bmatrix} 11 & -2 & 3 \\ 4 & 0 & -4 \\ -7 & 2 & 1 \end{bmatrix}$$

Now

$$A\,(\text{adj }A) = \begin{bmatrix}1&1&1\\3&4&7\\1&-1&1\end{bmatrix}\begin{bmatrix}11&-2&3\\4&0&-4\\-7&2&1\end{bmatrix}$$

$$= \begin{bmatrix}11+4-7 & -2+0+2 & 3-4+1\\33+16-49 & -6+0+14 & 9-16+7\\11-4-7 & -2+0+2 & 3+4+1\end{bmatrix}$$

$$= \begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix} \quad \ldots (1)$$

$$(\text{adj }A)\,A = \begin{bmatrix}11&-2&3\\4&0&-4\\-7&2&1\end{bmatrix}\begin{bmatrix}1&1&1\\3&4&7\\1&-1&1\end{bmatrix}$$

$$= \begin{bmatrix}11-6+3 & 11-8-3 & 11-14+3\\4+0-4 & 4+0+4 & 4+0-4\\-7+6+1 & -7+8-1 & -7+14+1\end{bmatrix} = \begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix} \ldots (2)$$

$$|A|.\,I_3 = 8\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix} \quad \ldots (3)$$

From (1), (2) and (3) $A\,(\text{adj }A) = (\text{adj }A)\,A = |A|\,I_3$.

6. **If** $A = \begin{bmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{bmatrix}$ **then, find the Inverse of A.**

Sol : $|A| = \begin{vmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{vmatrix}$

$$= 3\begin{vmatrix}6&-5\\-2&2\end{vmatrix} + 1\begin{vmatrix}-15&-5\\5&2\end{vmatrix} + 1\begin{vmatrix}-15&6\\5&-2\end{vmatrix}$$

$= 3\,(12-10) + 1\,(-30+25) + 1\,(30-30)$

$= 6-5 = 1 \neq 0 \qquad \therefore A^{-1}$ exists.

$$\text{adj }A = \begin{bmatrix} +\begin{vmatrix}6&-5\\-2&2\end{vmatrix} & -\begin{vmatrix}-15&-5\\5&2\end{vmatrix} & +\begin{vmatrix}-15&6\\5&-2\end{vmatrix}\\ -\begin{vmatrix}-1&1\\-2&2\end{vmatrix} & +\begin{vmatrix}3&1\\5&2\end{vmatrix} & -\begin{vmatrix}3&-1\\5&-2\end{vmatrix}\\ +\begin{vmatrix}-1&1\\6&-5\end{vmatrix} & -\begin{vmatrix}3&1\\-15&-5\end{vmatrix} & +\begin{vmatrix}3&-1\\-15&6\end{vmatrix}\end{bmatrix}^T$$

$$= \begin{bmatrix}+(12-10) & -(-30+25) & +(30-30)\\-(-2+2) & +(6-5) & -(-6+5)\\+(5-6) & -(-15+15) & +(18-15)\end{bmatrix}^T$$

$$= \begin{bmatrix}2&5&0\\0&1&1\\-1&0&3\end{bmatrix}^T = \begin{bmatrix}2&0&-1\\5&1&0\\0&1&3\end{bmatrix}$$

Now, $A^{-1} = \dfrac{1}{|A|}\text{adj }A$

$$= \frac{1}{1}\begin{bmatrix}2&0&-1\\5&1&0\\0&1&3\end{bmatrix} = \begin{bmatrix}2&0&-1\\5&1&0\\0&1&3\end{bmatrix}$$

7. **If** $A = \begin{bmatrix}1&-1\\2&3\end{bmatrix}$ **show that** $A^2 - 4A + 5I_2 = 0$ **and also find** A^{-1}.

Sol : $A = \begin{bmatrix}1&-1\\2&3\end{bmatrix}$

$$A^2 = \begin{bmatrix}1&-1\\2&3\end{bmatrix}\begin{bmatrix}1&-1\\2&3\end{bmatrix} = \begin{bmatrix}1-2 & -1-3\\2+6 & -2+9\end{bmatrix} = \begin{bmatrix}-1&-4\\8&7\end{bmatrix}$$

LHS $= A^2 - 4A + 5I_2$

$$= \begin{bmatrix}-1&-4\\8&7\end{bmatrix} - 4\begin{bmatrix}1&-1\\2&3\end{bmatrix} + 5\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$= \begin{bmatrix}-1&-4\\8&7\end{bmatrix} - \begin{bmatrix}4&-4\\8&12\end{bmatrix} + \begin{bmatrix}5&0\\0&5\end{bmatrix}$$

$$= \begin{bmatrix}-1&-4\\8&7\end{bmatrix} - \begin{bmatrix}1&4\\-8&-7\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} = 0 = \text{RHS}$$

$A^{-1} = \dfrac{1}{|A|}\text{adj }A$

$|A| = 3+2 = 5 \neq 0 \Rightarrow A^{-1} = \dfrac{1}{5}\begin{bmatrix}3&1\\-2&1\end{bmatrix}$

8. **Solve by using matrix inversion method:**
$x - y + z = 2,\ 2x - y = 0,\ 2y - z = 1.$

Sol : The given equations can be written in matrix form as

$$\begin{bmatrix}1&-1&1\\2&-1&0\\0&2&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix} \Rightarrow AX = B$$

Where $A = \begin{bmatrix}1&-1&1\\2&-1&0\\0&2&-1\end{bmatrix},\ X = \begin{bmatrix}x\\y\\z\end{bmatrix},\ B = \begin{bmatrix}2\\0\\1\end{bmatrix}$

$$|A| = \begin{vmatrix}1&-1&1\\2&-1&0\\0&2&-1\end{vmatrix}$$

$$= 1\begin{vmatrix}-1&0\\2&-1\end{vmatrix} + 1\begin{vmatrix}2&0\\0&-1\end{vmatrix} + 1\begin{vmatrix}2&-1\\0&2\end{vmatrix}$$

$$= 1(1-0)+1(-2-0)+1(4-0)$$
$$= 1-2+4 = 3 \neq 0$$

$\therefore$ A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (1-0) & -(-2-0) & +(4-0) \\ -(1-2) & +(-1-0) & -(2-0) \\ +(0+1) & -(0-2) & +(-1+2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

$$\text{Now } X = A^{-1}B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2+0+1 \\ 4+0+2 \\ 8+0+1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \therefore x = 1, y = 2, z = 3$$

9. The cost of 2 Kg of Wheat and 1 Kg of Sugar is ₹70. The cost of 1 Kg of Wheat and 1 Kg of Rice is ₹70. The cost of 3 Kg of Wheat, 2 Kg of Sugar and 1 Kg of rice is ₹170. Find the cost of per kg each item using matrix inversion method.

Sol : Let the cost of 1 kg of wheat, sugar and rice be x, y, z respectively.

By given data,

$$2x + y = 70$$
$$x + z = 70$$
$$3x + 2y + z = 170$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 70 \\ 70 \\ 170 \end{bmatrix} \Rightarrow AX = B$$

$$\text{Where } A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 70 \\ 70 \\ 170 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$

$$= 2(0-2) - 1(1-3) + 0 = -4 + 2 + 0 = -2 \neq 0$$

$\therefore A^{-1}$ exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0-2) & -(1-3) & +(2-0) \\ -(1-0) & +(2-0) & -(4-3) \\ +(1-0) & -(2-0) & +(0-1) \end{bmatrix}^T = \begin{bmatrix} -2 & 2 & 2 \\ -1 & 2 & -1 \\ 1 & -2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{-1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 70 \\ 70 \\ 170 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -140-70+170 \\ 140+140-340 \\ 140-70-170 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -40 \\ -60 \\ -100 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 50 \end{bmatrix}$$

$\therefore$ Cost of 1 kg of wheat is ₹20.

Cost of 1 kg of Sugar is ₹30.

Cost of 1 kg of Rice is ₹50.

10. The data are about an economy of two industries A and B. The values are in crores of rupees.

Producer	User		Final demand	Total output
	A	B		
A	50	75	75	200
B	100	50	50	200

Find the output when the final demand changes to 300 for A and 600 for B.

Sol :

$$a_{11} = 50,\ a_{12} = 75,\ x_1 = 200$$

$$a_{21} = 100,\ a_{22} = 50,\ x_2 = 200$$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{50}{200} = \frac{1}{4};\ b_{12} = \frac{a_{12}}{x_2} = \frac{\cancel{75}}{\cancel{200}} = \frac{3}{8}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{100}{200} = \frac{1}{2};\ b_{22} = \frac{a_{22}}{x_2} = \frac{\cancel{50}}{\cancel{200}} = \frac{1}{4}$$

∴ The technology matrix is $B = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-3}{8} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} \frac{3}{4} & \frac{-3}{8} \\ \frac{-1}{2} & \frac{3}{4} \end{vmatrix} = \left[\frac{3}{4}\right]\left[\frac{3}{4}\right] - \left[\frac{1}{2}\right]\left[\frac{3}{8}\right]$$

$$= \frac{9}{16} - \frac{3}{16} = \frac{\cancel{6}}{\cancel{16}} = \frac{3}{8} > 0$$

Since the diagonal elements of (I – B) are positive and |I – B| is positive, the system is viable

$$(I-B)^{-1} = \frac{1}{|I - B|}$$

$$\text{adj } (I - B) = \frac{1}{\frac{3}{8}} \begin{bmatrix} \frac{3}{4} & \frac{3}{8} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} = \frac{8}{3} \begin{bmatrix} \frac{3}{4} & \frac{3}{8} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 6 & 3 \\ 4 & 6 \end{bmatrix}$$

$$\text{Now, } X = (I - B)^{-1} D \text{ Where } D = \begin{bmatrix} 300 \\ 600 \end{bmatrix}$$

$$\therefore X = \frac{1}{3} \begin{bmatrix} 6 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 300 \\ 600 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1800 + 1800 \\ 1200 + 3600 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3600 \\ 4800 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1600 \end{bmatrix}$$

∴ The output is ₹1200 crores for A and ₹1600 crores for B.

Govt. Exam Questions & Answers

1 Marks

1. The value of $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ is **[GMQP-2019]**

(a) 0 (b) 1

(c) $(a - b)(b - c)(c - a)$ (d) 2 **Ans: (a) 0**

Hint: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \frac{1}{abc} \begin{vmatrix} a & a^2 & -abc \\ b & b^2 & -abc \\ c & c^2 & -abc \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$- \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

2. If n is the order of the matrix A, then |adjA| = **[Qy. & Hy. & Mar. 2019]**

(a) $|A|$ (b) $|A|^{n-1}$

(c) $|A|^{n+1}$ (d) $|A|^n$ **Ans: (b) $|A|^{n-1}$**

3. The value of x if $\begin{vmatrix} x & x+2 \\ x-2 & x \end{vmatrix}$ is : **[Mar. - 2020]**

(a) x^2 (b) + 4 (c) 0 (d) 1

Ans: (c) 0

4. The sum of the diagonal elements of a square matrix is :

(a) Trace (b) Zero **[Sep. - 2020]**

(c) Triangular (d) All the above **Ans: (a) Trace**

5. If A is 3 × 3 matrix and |A| = 16, then $|A^{-1}|$ is equal to _____. **[CRT - 2022]**

(a) $\frac{1}{4}$ (b) $\frac{1}{16}$ (c) 2 (d) 4

Ans: (b) $\frac{1}{16}$

6. If $\begin{vmatrix} x & 3 \\ 4 & 5 \end{vmatrix} = 0$ then the value of x is : **[Qy. - 2023]**

(a) $\frac{-12}{5}$ (b) $\frac{5}{12}$ (c) $\frac{12}{5}$ (d) $\frac{-5}{12}$

Ans: (c) $\frac{12}{5}$

7. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then A (Adj A) = **[Hy. - 2023]**

(a) I_2 (b) I_4 (c) I_3 (d) |A|

Ans: (a) I_2

2 Marks

1. Solve $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$ **[GMQP-2019]**

Sol : $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

$\Rightarrow (x-1)(x-2)(x-3) = 0x = 1, x = 2, x = 3$

2. Evaluate : $\begin{vmatrix} x & x+2 \\ x-2 & x \end{vmatrix}$ **[Hy.-2019]**

Sol : $\begin{vmatrix} x & x+2 \\ x-2 & x \end{vmatrix} = x^2 - (x-2)(x+2)$

$= x^2 - (x^2 - 2^2) = \cancel{x^2} - \cancel{x^2} + 4$

$\begin{vmatrix} x & x+2 \\ x-2 & x \end{vmatrix} = 4$

3. Show that $\begin{vmatrix} x & y & z \\ 2x+2z & 2y+2b & 2z+2c \\ a & b & c \end{vmatrix} = 0.$ **[Mar-2019]**

Sol : $\begin{vmatrix} x & y & z \\ 2x+2z & 2y+2b & 2z+2c \\ a & b & c \end{vmatrix}$

$= \begin{vmatrix} x & y & z \\ 2x & 2y & 2z \\ a & b & c \end{vmatrix} + \begin{vmatrix} x & y & z \\ 2a & 2b & 2c \\ a & b & c \end{vmatrix} = 0 + 0 = 0$

4. Construct 3 × 2 matrix whose elements are $a_{ij} = |2i - j|$. **[Sep.-2020]**

Sol : $a_{ij} = |2i - j|$

The required 3 × 2 matrix is $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

$a_{11} = |2(1) - 1| = |2 - 1| = 1$

$a_{12} = |2(1) - 2| = |2 - 2| = 0$

$a_{21} = |2(2) - 1| = |4 - 1| = 3$

$a_{22} = |2(2) - 2| = |4 - 2| = 2$

$a_{31} = |2(3) - 1| = |6 - 1| = 5$

$a_{32} = |2(3) - 2| = |6 - 2| = 4$

$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{pmatrix}$

5. If $A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$ then, find A^{-1}. **[May - 2022]**

Sol : $A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 4 \\ -3 & 2 \end{vmatrix} = 16 \neq 0$

Since A is a nonsingular matrix, A^{-1} exists.

Now adj A $= \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{16}\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$

6. If $A = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix}$, then find A^{-1}. **[CRT - 2022]**

Sol : Refer Textbook Example 1.15

7. Show that $\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$ is non - singular. **[July - 2023]**

Sol : Let A $= \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$

$|A| = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$

$= 24 - 8 = 16 \neq 0$

$\therefore$ A is a non-singular matrix

8. If $A = \begin{bmatrix} 2 & 4 \\ -5 & 2 \end{bmatrix}$ then, find A^{-1}. **[Qy. - 2023]**

Sol : $A = \begin{vmatrix} 2 & 4 \\ -5 & 2 \end{vmatrix}$

$|A| = \begin{vmatrix} 2 & 4 \\ -5 & 2 \end{vmatrix} = 4 - (-20)$

$= 4 + 20 = 24 = 24 \neq 0$

Since A is a non - singular matrix, A^{-1} exists

Now, adj A $= \begin{vmatrix} 2 & -4 \\ 5 & 2 \end{vmatrix}$

$A^{-1} = \frac{1}{|A|}$ adj $A = \frac{1}{24}\begin{vmatrix} 2 & -4 \\ 5 & 2 \end{vmatrix}$

9. Find x and y if $x + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $x - y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ **[Hy. - 2023]**

Sol : $\overset{(1)}{x + \cancel{y}} + \overset{(2)}{x - \cancel{y}} = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$2x = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$

$$x = \begin{bmatrix} \frac{10}{2} & \frac{0}{2} \\ \frac{2}{2} & \frac{8}{2} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

$$x + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$$

$$y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

10. Evaluate $\begin{vmatrix} 1 & 3 & 4 \\ 102 & 18 & 36 \\ 17 & 3 & 6 \end{vmatrix}$ **[Mar. - 2024]**

Sol : Refer Textbook Example 1.8

3 Marks

1. If A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ then show that $(AB)^{-1} = B^{-1}A^{-1}$. **[Qy.-2019]**

Sol : $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$|A| = -2 \quad = \quad |B| = 1$$

$$AB = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \quad |AB| = -2$$

$$\text{adj}(AB) = \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{2}\begin{pmatrix} -5 & 3 \\ 4 & 2 \end{pmatrix}$$

$$B^{-1}A^{-1} = \frac{-1}{2}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{pmatrix} -5 & 3 \\ 4 & 2 \end{pmatrix}$$

2. For A = $(a_{ij})_{2 \times 2}$ defined by $a_{ij} = 2i - j$. Prove that $AA^{-1} = 1$. **[Mar. - 2019]**

Sol : $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$

$$|A| = 2$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$$

$$AA^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

3. If A = $\begin{bmatrix} 5 & -7 \\ 3 & -4 \end{bmatrix}$ then prove that $(A^{-1})^{-1} = A$. **[Mar.- 2020]**

Sol :

$$|A| = -20 - (-21) = -20 + 21 = 1 = 1$$

$$A^{-1} = \begin{bmatrix} -4 & 7 \\ -3 & 5 \end{bmatrix}$$

$$|A^{-1}| = -20 - (-21) = -20 + 21 = 1$$

$$(A^{-1})^{-1} = \begin{bmatrix} 5 & -7 \\ 3 & -4 \end{bmatrix} = A$$

4. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$. Test whether the system is viable as per Hawkins - Simon conditions. **[CRT - 2022; Mar. - 2024]**

Sol : Refer Textbook Example 1.23

5. Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$. **[Aug. - 2022; Mar. - 2023]**

Sol : Refer Textbook Example 1.6

6. Solve by using matrix inversion method :
$2x + 5y = 1$; $3x + 2y = 7$ **[July & Hy. - 2023]**

Sol : The given system can be written as

$$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\text{i.e, } AX = B$$

$$X = A^{-1}B$$

$$\text{Where } A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = -11 \neq 0$$

A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{-11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$x = 3$ and $y = -1$.

5 Marks

1. In an economy, there are two industries P_1 and P_2 and the following table gives the supply and the demand position in crores of rupees.

Production Sector	Consumption Sector		Final Demand	Total output
	P_1	P_2		
P_1	10	25	15	50
P_2	20	30	10	60

Determine the outputs when the final demand 35 for P_1 and 42 for P_2.

(i) Write the technology matrix

(ii) Test Hawkin's - Simon conditions for the viability of the system.

[First Mid - 2018; Mar. - 2019, Sep. - 2020; July - 2023]

Sol : With the usual notation we have,

$a_{11} = 10, a_{12} = 25, x_1 = 50$

$a_{21} = 20, a_{22} = 30, x_2 = 60$

Now, $b_{11} = \frac{a_{11}}{x_1} = \frac{1\cancel{0}}{5\cancel{0}} = \frac{1}{5}$, $b_{12} = \frac{a_{12}}{x_2} = \frac{\cancel{25}^{5}}{\cancel{60}_{12}} = \frac{5}{12}$

$b_{21} = \frac{a_{21}}{x_1} = \frac{2\cancel{0}}{5\cancel{0}} = \frac{2}{5}$, $b_{22} = \frac{a_{22}}{x_2} = \frac{\cancel{30}^{1}}{\cancel{60}_{2}} = \frac{1}{2}$

∴ The technology matrix is

$$B = \begin{bmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{bmatrix}$$

$$I - B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{pmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{bmatrix}$$

$$\therefore |I - B| = \begin{vmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{vmatrix} = \left(\frac{4}{5}\right)\left(\frac{1}{2}\right) - \left(\frac{-2}{5}\right)\left(\frac{-5}{12}\right) = \frac{7}{30}$$

$$\therefore (I - B)^{-1} = \frac{1}{\frac{7}{30}} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{30}{7} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix}$$

Now, $X = (I - B)^{-1}D$

$$= \frac{1}{7} \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix} \begin{pmatrix} 35 \\ 42 \end{pmatrix} = \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 75 + 75 \\ 60 + 144 \end{pmatrix} = \begin{pmatrix} 150 \\ 204 \end{pmatrix}$$

The output of the industry P_1 should be ₹ 150 crores and that of P_2 should be ₹ 204 crores.

2. The cost of 4 kg onion, 3 kg what and 2 kg rice is ₹320. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹560. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 380. Find the cost of each item per kg matrix inversion method. **[June - 2019; Mar. - 2023]**

Sol : Let x, y and z be the cost of onion, wheat and rice per kg respectively.

Then,
$$4x + 3y + 2z = 320$$
$$2x + 4y + 6z = 560$$
$$6x + 2y + 3z = 380.$$

It can be written as

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$AX = B$$

where $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} = 50 \neq 0$$

A^{-1} exist

$A_{11} = 0$ $\quad A_{12} = 30$ $\quad A_{13} = -20$

$A_{21} = -5$ $\quad A_{22} = 0$ $\quad A_{23} = 10$

$A_{31} = 10$ $\quad A_{32} = -20$ $\quad A_{33} = 10$

$$[A_{ij}] = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$A-1 = \frac{1}{|A|} adj A = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$X = A^{-1} B = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$= \frac{1}{5} \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 32 \\ 56 \\ 38 \end{bmatrix} = \begin{bmatrix} 0 - 280 + 380 \\ 960 + 0 - 760 \\ -640 + 560 + 380 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$$

$x = 20$, $y = 40$ and $z = 60$.

Cost of 1 kg onion is ₹ 20, cost of 1 kg wheat is ₹ 40 and cost of 1 kg rice is ₹ 60.

3. Show that the matrices A = $\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{pmatrix}$ are inverse of each other. **[Qy. - 2019; May - 2022]**

Sol : $AB = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{pmatrix}$

$$= \frac{1}{35}\begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

III^{ly}, BA = I

$\therefore$ AB = BA = I

$\therefore$ A and B are inverse of each other.

4. Evaluate $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ **[Mar. - 2020; CRT - 2022]**

Sol : Consider $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} =$

$$\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} R_1 \to R_1 \to R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$$

Taking $(a-b)(b-c)$ common from R_1 and R_2,

$$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$= (a-b)(b-c)[0-0+\{b+c-(a+b)\}]$ [Expanding along C_1]

$= (a-b)(b-c)(c-a)$

5. If A = $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$, then show that $(AB)^{-1} = B^{-1}A^{-1}$. **[May - 2022]**

Sol : $|A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-2 = -1 \neq 0$

$\therefore A^{-1}$ exists.

$|B| = \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 0+1 = 1 \neq 0$

$\therefore B^{-1}$ also exists.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$$

$\therefore (AB)^{-1}$ exists.

$$\text{adj}(AB) = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|}\text{adj}(AB) = \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \quad \text{... (1)}$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}adj\, B = \frac{1}{1}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$

Hence proved.

6. **If** $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ **satisfies the equation** $A^2 - kA + I_2 = 0$ **then, find** k **and also** A^{-1}. **[CRT - 2022]**

Sol : Refer Textbook Example 1.17

7. **Solve by using matrix inversion method** $2x + y = 70,\ x + z = 70,\ 3x + 2y + z = 170$. **[Qy. - 2023]**

Sol :

$$\begin{aligned} 2x + y &= 70 \\ x + z &= 70 \\ 3x + 2y + z &= 170 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 70 \\ 70 \\ 170 \end{bmatrix} \Rightarrow AX = B$$

$$\text{Where } A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$B = \begin{bmatrix} 70 \\ 70 \\ 170 \end{bmatrix} \Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$

$$= 2(0-2) - 1(1-3) + 0$$

$$= -4 + 2 + 0 = -2 \neq 0$$

$\therefore A^{-1}$ exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0-2) & -(1-3) & +(2-0) \\ -(1-0) & +(2-0) & -(4-3) \\ +(1-0) & +(2-0) & +(0-1) \end{bmatrix}^T = \begin{bmatrix} -2 & 2 & 2 \\ -1 & 2 & -1 \\ 1 & -2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{-1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 70 \\ 70 \\ 170 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -140 - 70 + 170 \\ 140 + 140 - 340 \\ 140 - 70 - 170 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -40 \\ -60 \\ -100 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 50 \end{bmatrix}$$

8. **An economy produces only coal and steel. These two commodities serve as intermediate inputs in each other's production. 0.4 tonne of steel and 0.7 tonne of coal are needed to produce a tonne of steel, Similarly 0.1 tonne of steel and 0.6 tonne of coal are required to produce a tonne of coal. No capital inputs are needed. Do you think that the system is viable? 2 and 5 labour days are required to produce a tonnes of coal and steel respectively. If economy needs 100 tonnes of coal and 50 tonnes of steel, calculate the gross output of the two commodities and the total labour days required.** **[Mar. - 2024]**

Sol : Refer Textbook Example 1.26

ADDITIONAL PROBLEMS

1 MARK

FIND THE ODD ONE OUT OF THE FOLLOWING

1. (a) $\begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}$ (d) $\begin{pmatrix} \sin a & \cos a \\ -\cos a & \sin a \end{pmatrix}$

Ans: (b) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

Hint: (a), (c) and (d) are nonsingular and (b) is singular.

2. (a) Minor (b) Cofactor
(c) Adjoint (d) Determinant

Ans: (c) Adjoint

Hint: (a), (b) and (d) represent number and (c) represent matrix.

Choose the incorrect pair.

1.

(a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	=	$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$, where M_{ij} is minor of a_{ij}.
(b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	=	$a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$, where A_{ij} is minor of a_{ij}.
(c) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	=	$a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$
(d) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	=	$a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31}$

Ans: (c) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$

Hint: Expanding along first column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$$

Assertion & Reason :

1. **Assertion (A)** : The inverse of does not exist. $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$

Reason (R) : The matrix in non singular.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not a correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

Ans: (a) Both A and R are true and R is the correct explanation of A

2. **Assertion (A)** : A is a square matrix of order 3, then $|kA|$ is k^3A.

Reason (R) : Multiplying a row by 3 result in 3A. There are 3 rows in A.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not a correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

Ans: (a) Both A and R are true and R is the correct explanation of A

Low Order Thinking Skills (LOTS)

1. **If $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ and $B = [-1 \quad 2 \quad 1]$, verify that $(AB)^T = B^T \cdot A^T$**

Sol :

$$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}[-1 \quad 2 \quad 1] = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

$$\therefore (AB)^T = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad ...(1)$$

$$B^T = [-1 \quad 2 \quad 1]^T = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{and } A^T = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}^T = [1 \quad -4 \quad 3]$$

$$\therefore B^TA^T = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}[1 \quad -4 \quad 3] = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad ...(2)$$

From (1) and (2), $(AB)^T = B^T \cdot A^T$.

2. **If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ then show that $|2A| = 4|A|$.**

Sol :

Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then $2A = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$$\therefore |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 8 - 32 = -24 \quad ...(1)$$

$$\text{Also, } |A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 2 - 8 = -6$$

$$\therefore 4|A| = 4(-6) = -24 \quad ...(2)$$

From (1) and (2),

$$|2A| = 4 \cdot |A|$$

3. **Evaluate $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$.**

Sol :

$$\text{Let } |A| = \begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

Expanding along R_1 we get,

$$|A| = 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$$

$$= 2(0-5) + 1(0+3) - 2(0-6)$$

$$= -10 + 3 + 12 = 5$$

$$\therefore |A| = 5$$

4. Find the values of x if $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$.

Sol : Given $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow \quad 2 - 20 = 2x^2 - 24$

$\Rightarrow \quad -18 = 2x^2 - 24$

$\Rightarrow \quad -18 + 24 = 2x^2$

$\Rightarrow \quad 6 = 2x^2$

$\Rightarrow \quad x^2 = 3$

$\Rightarrow \quad x = \pm\sqrt{3}$

5. Using the property of determinant, evaluate $\begin{vmatrix} 6 & 5 & 12 \\ 2 & 4 & 4 \\ 2 & 1 & 4 \end{vmatrix}$

Sol : Let $|A| = \begin{vmatrix} 6 & 5 & 12 \\ 2 & 4 & 4 \\ 2 & 1 & 4 \end{vmatrix}$

Taking 2 common from C_1 and 4 common from C_3, we get,

$$|A| = 2 \times 4\begin{vmatrix} 3 & 5 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 8 \times 0 \quad [\because C_1 \equiv C_3] = 0$$

6. Using the property of determinants show that $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$.

Sol : Let $A = \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$

Applying the elementary transformation, $C_1 \to C_1 + C_2$ we get,

$$A = \begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix} = 0 \; [C_1 \equiv C_3]$$

$$\therefore |A| = 0$$

7. Write the minors and co-factors of the elements of $\begin{vmatrix} 5 & 3 \\ -6 & 2 \end{vmatrix}$.

Sol : Let $A = \begin{vmatrix} 5 & 3 \\ -6 & 2 \end{vmatrix}$

Minor of 5 $= M_{11} = 2$ and $A_{11} = (-1)^{1+1} M_{11} = 2$

Minor of 3 $= M_{12} = -6$ and $A_{12} = (-1)^{1+2} M_{12} = 6$

Minor of -6 $= M_{21} = 3$ and $A_{21} = (-1)^{2+1} M_{21} = -3$

Minor of 2 $= M_{22} = 5$ and $A_{22} = (-1)^{2+2} M_{22} = 5$

8. Verify that A(adj A) = (adj A) A = |A|I for the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Sol : Given $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 8 + 3 = 11$$

Now, $A_{11} = 4$, $A_{12} = -(-1) = 1$, $A_{21} = -3$, $A_{22} = 2$

$$\therefore \text{adj } A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$\therefore A(\text{adj } A) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+3 & -6+6 \\ -4+4 & 3+8 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = 11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A| I_2 \quad \text{...(1)}$$

Also $\quad (\text{adj } A) A = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 8+3 & 12-12 \\ 2-2 & 3+8 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = 11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A| . I_2 \quad \text{...(2)}$$

From (1) and (2), A(adj A) = (adj A) A = |A| . I_2

9. Find the inverse of $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$.

Sol : Let $A = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix}$

$$\therefore |A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2 + 15 = 13$$

Now, $A_{11} = 2$, $A_{12} = -(-3) = 3$, $A_{21} = -5$, $A_{22} = -1$

$$\therefore \text{adj } A = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

$$\text{Now } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

10. Solve: $2x + 5y = 1$ and $3x + 2y = 7$ using matrix method.

Sol : Given equations are $2x + 5y = 1$; $3x + 2y = 7$

This system of equations can be written in matrix form as $\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \Rightarrow AX = B$

$$\text{where } A = \begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\therefore X = A^{-1}B.$$

$$|A| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = 4 - 15 = -11$$

$$A_{11} = 2, A_{12} = -3, A_{21} = -5, A_{22} = 2$$

$$\therefore \text{adj } A = \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{-1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B = \frac{-1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= \frac{-1}{11}\begin{bmatrix} +2-35 \\ -3+14 \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\therefore x = 3 \text{ and } y = -1.$$

11. The data below are about an economy of two industries P and Q. The values are in lakhs of rupees.

Producer	User P	User Q	Final Demand	Total output
P	16	12	12	40
Q	12	8	4	24

Find the technology matrix and check whether the system is viable as per Hawkins-Simon conditions.

Sol : With the usual notation we have

$$a_{11} = 16, a_{12} = 12, x_1 = 40$$

$$a_{21} = 12, a_{22} = 8, x_2 = 24$$

$$\text{Now } b_{11} = \frac{a_{11}}{x_1} = \frac{\cancel{16}^{2}}{\cancel{40}_{5}} = \frac{2}{5},\ b_{12} = \frac{a_{12}}{x_2} = \frac{\cancel{12}^{1}}{\cancel{24}_{2}} = \frac{1}{2}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{\cancel{12}^{3}}{\cancel{40}_{10}} = \frac{3}{10},\ b_{22} = \frac{a_{22}}{x_2} = \frac{\cancel{8}^{1}}{\cancel{24}_{3}} = \frac{1}{3}$$

The technology matrix is

$$B = \begin{bmatrix} \frac{2}{5} & \frac{1}{2} \\ \frac{3}{10} & \frac{1}{3} \end{bmatrix}$$

$$I - B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{5} & \frac{1}{2} \\ \frac{3}{10} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{2} \\ -\frac{3}{10} & \frac{2}{3} \end{pmatrix}$$

The main diagonal elements in $(I - B)$ namely $\frac{3}{5}$ and $\frac{2}{3}$ are positive.

$$\text{Also, } |I - B| = \begin{vmatrix} \frac{3}{5} & \frac{-1}{2} \\ \frac{-3}{10} & \frac{2}{3} \end{vmatrix} = \left(\frac{3}{5}\right)\left(\frac{2}{3}\right) - \left(\frac{1}{2}\right)\left(\frac{3}{10}\right)$$

$$= \frac{2}{5} - \frac{3}{20} = \frac{8-3}{20} = \frac{\cancel{5}^{1}}{\cancel{20}_{4}} = \frac{1}{4}$$

$\therefore$ $|I - B|$ is positive.

$\therefore$ The two Hawkin's Simon conditions are satisfied. Hence the system is viable.

Middle Order Thinking Skills (MOTS)

1. Show that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

Sol :

$$\text{LHS} = \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ we get,

$$\text{LHS} = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix}$$

Taking $(x + a + b + c)$ common from C_1 we get,

$$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get,

$$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along C_1 we get,

$$= (x+a+b+c)\left[1\begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix} + 0 + 0\right]$$

$$= (x + a + b + c)(x^2) = \text{RHS}$$

Hence proved.

2. **Show that** $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$

Sol : LHS $= \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ we get,

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Expanding along C_1 we get

$$= a\begin{vmatrix} a & 2a+b \\ 3a & 7a+3b \end{vmatrix} - 0 + 0$$

$$= a(7a^2 + \cancel{3ab} - 6a^2 - \cancel{3ab})$$

$$= a(7a^2 - 6a^2) = a(a^2) = a^3 = \text{RHS}$$

Hence proved.

3. **Using the properties of determinants, show that** $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0.$

Sol : Let A $= \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

Applying $C_1 \to C_1 + 9C_2$ we get,

$$A = \begin{vmatrix} 2+63 & 7 & 65 \\ 3+72 & 8 & 75 \\ 5+81 & 9 & 86 \end{vmatrix} = \begin{vmatrix} 65 & 7 & 65 \\ 75 & 8 & 75 \\ 86 & 9 & 86 \end{vmatrix} = 0 \; [\because C_1 \equiv C_3]$$

4. **Using co-factors of elements of second column evaluate** $\begin{vmatrix} 6 & -1 & 5 \\ 3 & 0 & 4 \\ -2 & 7 & -3 \end{vmatrix}.$

Sol : Let $\Delta = \begin{vmatrix} 6 & -1 & 5 \\ 3 & 0 & 4 \\ -2 & 7 & -3 \end{vmatrix}$

Now $M_{12} = \begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix} = -9 - (-8) = -1$

$M_{22} = \begin{vmatrix} 6 & 5 \\ -2 & -3 \end{vmatrix} = -18 - (-10) = -8$

$M_{32} = \begin{vmatrix} 6 & 5 \\ 3 & 4 \end{vmatrix} = 24 - 15 = 9$

Now expansion of |A| using co-factors of elements of second column we get,

$|A| = a_{12} A_{12} + a_{21} A_{21} + a_{31} A_{31}$

$A_{12} = (-1)^{1+2} M_{12} = (-1)(-1) = 1$

$A_{22} = (-1)^{2+2} M_{22} = 1(-8) = -8$

$A_{32} = (-1)^{3+2} M_{32} = -(9) = -9$

$\therefore |A| = -1(1) + 0(-8) + 7(-9) = -1 - 63 = -64.$

5. **Find the adjoint of the matrix** $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 1 \\ 3 & 6 & 8 \end{bmatrix}.$

Sol : Let A $= \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 1 \\ 3 & 6 & 8 \end{bmatrix}$

$A_{11} = 40 - 6 = 34,$

$A_{12} = -(0 - 3) = 3,$

$A_{13} = 0 - 15 = -15$

$A_{21} = -(-8 - 18) = 26,$

$A_{22} = 16 - 9 = 7,$

$A_{23} = -(12 + 3) = -15$

$A_{31} = -1 - 15 = -16,$

$A_{32} = -(2 - 0) = -2,$

$A_{33} = 10 - 0 = 10$

$$\therefore \text{adj } A = \begin{bmatrix} 34 & 3 & -15 \\ 26 & 7 & -15 \\ -16 & -2 & 10 \end{bmatrix}^T = \begin{bmatrix} 34 & 26 & -16 \\ 3 & 7 & -2 \\ -15 & -15 & 10 \end{bmatrix}$$

6. **If** $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$,

Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol : Given A $= \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1 \Rightarrow A^{-1}$ exists.

$A_{11} = 5, A_{12} = -7, A_{21} = -2, A_{22} = 3$

$$\therefore \text{adj A} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj A}$$

$$= \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$\text{Now B} = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 4 & 6 \\ 3 & 2 \end{vmatrix} = 8 - 18 = -10 \Rightarrow B^{-1} \text{ exists.}$$

$$B_{11} = 2, B_{12} = -3, B_{21} = -6, B_{22} = 4$$

$$\therefore \text{adj B} = \begin{bmatrix} 2 & -3 \\ -6 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{|B|}\text{ adj B}$$

$$= -\frac{1}{10}\begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -2 & 6 \\ 3 & -4 \end{bmatrix}$$

$$\therefore B^{-1}A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 6 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -10-42 & 4+18 \\ 15+28 & -6-12 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix} \quad \dots (1)$$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+6 & 18+4 \\ 28+15 & 42+10 \end{bmatrix} = \begin{bmatrix} 18 & 22 \\ 43 & 52 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 18 & 22 \\ 43 & 52 \end{vmatrix} = 936 - 946$$

$$= -10 \Rightarrow (AB)^{-1} \text{ exists.}$$

$$\text{adj AB} = \begin{bmatrix} 52 & -43 \\ -22 & 18 \end{bmatrix}^T = \begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|}\text{ adj (AB)}$$

$$= \frac{-1}{10}\begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix} \quad \dots (2)$$

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$.

Hence proved.

7. Find the numbers a and b such that $A^2 + aA + bI = 0$ for the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

Sol : Given $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Given $A^2 + aA + bI = 0$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 11+3a+b & 8+2a+0 \\ 4+a+0 & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the like terms we get,

$$4 + a = 0$$
$$\Rightarrow a = -4$$
$$3 + a + b = 0$$
$$\Rightarrow 3 - 4 + b = 0$$
$$\Rightarrow -1 + b = 0$$
$$\Rightarrow b = 1$$

8. Prove that $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

Sol : Let $A = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

Expanding along R_1 we get

$$|A| = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta)$$
$$+ \cos\theta(-\sin\theta + x\cos\theta)$$

$$= -x^3 - x + x\sin^2\theta + \cancel{\sin\theta\cos\theta} - \cancel{\sin\theta\cos\theta} + x\cos^2\theta$$

$$= -x^3 - x + x(\sin^2\theta + \cos^2\theta)$$

$$= -x^3 - x + x(1) \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$= -x^3$ which is independent of θ.

9. Using matrix method, solve $x + 2y + z = 7$, $x + 3z = 11$ and $2x - 3y = 1$.

Sol : The system of equations can be written in the form AX = B where,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\text{Now,} |A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix}$$

$$= 1\,(0+9) - 2\,(0-6) + 1\,(-3-0)$$

$$= 9 + 12 - 3 = 18 \neq 0 \Rightarrow A^{-1} \text{ exists.}$$

$A_{11} = 0 + 9 = 9,$

$A_{12} = -(0-6) = 6,\ A_{13} = -3 - 0 = -3$

$A_{21} = -(0+3) = -3,\ A_{22} = 0 - 2 = -2,$

$A_{23} = -(-3-4) = 7$

$A_{31} = 6 - 0 = 6,$

$A_{32} = -(3-1) = -2,$

$A_{33} = 0 - 2 = -2$

$$\therefore \text{adj } A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 63-33+6 \\ 42-22-2 \\ -21+77-2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore x = 2, y = 1,$ and $z = 3.$

10. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is such that $A^T = A^{-1}$, find α.

Sol : Given that $A^T = A^{-1}$

$\Rightarrow \quad AA^T = AA^{-1}$

$\Rightarrow \quad AA^T = I$

$$\text{Now, } AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cancel{-\sin\alpha\cos\alpha} + \cancel{\sin\alpha\cos\alpha} \\ \cancel{-\sin\alpha\cos\alpha} + \cancel{\sin\alpha\cos\alpha} & +\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [\because \sin^2\alpha + \cos^2\alpha = 1]$$

Thus, $AA^T = I$ is true for all α.

Hence α can take any real value.

HIGH ORDER THINKING SKILLS (HOTS)

1. Determine the values of x for which the matrix $A = \begin{bmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{bmatrix}$ is singular.

Sol : Given matrix A is singular, if $|A| = 0$

$$|A| = \begin{vmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0$$

Expanding along R_1 we get,

$$|A| = x+1\begin{vmatrix} x+2 & 2 \\ 1 & x-6 \end{vmatrix} + 3\begin{vmatrix} -5 & 2 \\ 4 & x-6 \end{vmatrix} + 4\begin{vmatrix} -5 & x+2 \\ 4 & 1 \end{vmatrix} = 0$$

$\Rightarrow (x+1)\,[(x+2)\,(x-6) - 2] + 3[-5\,(x-6) - 8]$

$\qquad + 4\,[-5 - 4\,(x+2)] = 0$

$\Rightarrow (x+1)\,[x^2 - 4x - 12 - 2] + 3[-5x + 30 - 8]$

$\qquad + 4\,[-5 - 4x - 8] = 0$

$\Rightarrow (x+1)\,(x^2 - 4x - 14) + 3(-5x + 22) + 4(-4x - 13) = 0$

$\Rightarrow x^3 - 4x^2 - 14x + x^2 - 4x - 14 - 15x + 66 - 16x - 52 = 0$

$\Rightarrow x^3 - 3x^2 - 49x = 0 \Rightarrow x(x^2 - 3x - 49) = 0$

$$\Rightarrow x = 0 \text{ or } x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-49)}}{2a}$$

$$\left[\because x = \frac{-b \pm \sqrt{b^2 - 4ab}}{2a}, a = 1, b = -3, c = -49\right]$$

$$\Rightarrow x = 0 \text{ or } x = \frac{3 \pm \sqrt{9 + 196}}{2} \Rightarrow x = 0 \text{ or } x = \frac{3 \pm \sqrt{205}}{2}$$

2. Without expanding show that

$$\Delta = \begin{vmatrix} \mathrm{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \mathrm{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Sol :

$$\text{Given } \Delta = \begin{vmatrix} \mathrm{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \mathrm{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_2$, we get,

$$\Delta = \begin{vmatrix} \mathrm{cosec}^2\theta - \cot^2\theta & \cot^2\theta & 1 \\ \cot^2\theta - \mathrm{cosec}^2\theta & \mathrm{cosec}^2\theta & -1 \\ 42-40 & 40 & 2 \end{vmatrix} = \begin{vmatrix} 1 & \cot^2\theta & 1 \\ -1 & \mathrm{cosec}^2\theta & -1 \\ 2 & 40 & 2 \end{vmatrix}$$

$[\because \mathrm{cosec}^2\theta - \cot^2\theta = 1]$

$= 0 \qquad [\because C_1 \equiv C_3]$

$\Delta = 0$

3. If a, b, c are in A.P, find the value of
$$\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}.$$

Sol : Let A $= \begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$

Applying $R_2 \to 2R_2$ and dividing by 2 we get,

$$A = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 6y+10 & 12y+16 & 18y+2b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

Applying $R_2 \to R_2 - (R_1 + R_3)$ we get

$$A = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 2b-(a+c) \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

Given a, b, c in A.P. $\Rightarrow 2b = a + c \Rightarrow 2b - (a + c) = 0$

$$\therefore A = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 0 \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} = 0$$

$\therefore |A| = 0$

4. Let a, b and c denote the sides BC, CA and AB respectively of Δ ABC. If $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then find the value of $\sin^2A + \sin^2B + \sin^2C$.

Sol : Give A $= \begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$A = \begin{vmatrix} 1 & a & b \\ 0 & c-a & a-b \\ 0 & b-a & c-b \end{vmatrix} = 0$$

Expanding along C_1 we get

$$1\begin{vmatrix} c-a & a-b \\ b-a & c-b \end{vmatrix} = 0$$

$\Rightarrow$ $(c-a)(c-b) - (b-a)(a-b) = 0$

$\Rightarrow$ $c^2 - bc - ac + ab - (ab - b^2 - a^2 + ab) = 0$

$\Rightarrow$ $a^2 + b^2 + c^2 - ab - bc - ca = 0$

Multiplying both sides by 2 we get,

$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow$ $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$\Rightarrow$ $a - b = 0,\ b - c = 0,\ c - a = 0$

$\Rightarrow$ $a = b = c \Rightarrow \Delta$ ABC is equilateral.

$\therefore$ A = B = C = $\frac{\pi}{3}$

$\therefore \sin^2A + \sin^2B + \sin^2C$

$$= 3\sin^2\frac{\pi}{3} = 3\left(\sin\frac{\pi}{3}\right)^2 = 3\left(\frac{\sqrt{3}}{2}\right)^2 = 3\times\frac{3}{4} = \frac{9}{4}$$

5. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Sol : $|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + \tan^2 x$

$= \sec^2 x \neq 0 \Rightarrow A^{-1}$ exists

Let C_{ij} be the co-factor of a_{ij} in A

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (1) = 1$

$C_{12} = (-1)^{1+2} (-\tan x) = \tan x$

$C_{21} = (-1)^{2+1} \tan x = -\tan x$

$C_{22} = (-1)^{2+2} (1) = 1$

$$\therefore \text{adj } A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$$

$$\therefore A^TA^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1+\tan^2 x} \frac{-\tan x}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} + \frac{\tan x}{1+\tan^2 x} & \frac{-\tan^2 x}{1+\tan^2 x} \frac{1}{1+\tan^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\tan x}{1+\tan^2 x} & \frac{-2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

(Using multiple angle formulae)