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PREFACE

Sir,
An equation has no meaning, for me
unless it expresses a thought of GOD

- Ramanujam
[Statement to a friend]

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers,

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **Sura's Mathematics Guide Volume I and Volume II** for +2 Standard – Edition 2021-22. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises and in-text questions in addition to precise answers for textual questions.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

Subash Raj, B.E., M.S.
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All the Best

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Volume - I

MATHEMATICS

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VOLUME - I

CHAPTER 1

APPLICATIONS OF MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

- If $|A| \neq 0$, then A is a non-singular matrix and if $|A| = 0$, then A is a singular matrix.
- The adjoint matrix of A is defined as the transpose of the matrix of co-factors of A.
- If $AB = BA = I_n$, then the matrix B is called inverse of A.
- If a square matrix has an inverse, then it is unique.
- A^{-1} exists if and only if A is non-singular.
- Singular matrix has no inverse.
- If A is non – singular and AB = AC, then B = C (left cancellation law).
- If A is non – singular and BA = CA then B = C (Right cancellation law).
- If A and B are any two non-singular square matrices of order n, then adj (AB) = (adj B) (adj A)
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- Two matrices A and B of same order are said to the **equivalent** if one can be obtained from the other by the applications of elementary transformations (A~B).
- A non – zero matrix is in a **row - echelon f**orm if all zero rows occur as bottom rows of the matrix and if the first non – zero element in any lower row occurs to the right of the first non – zero entry in the higher row.
- The **rank** of a matrix A is defined as the order of a highest order non – vanishing minor of the matrix A [ρ(A)].
- The **rank** of a non – zero matrix is equal to the number of non – zero rows in a row – echelon form of the matrix.
- An **elementary matrix** is a matrix which is obtained from an identity matrix by applying only one elementary transformation. Every non-singular matrix can be transformed to an identity matrix by a sequence of elementary row operations.
- A system of linear equations having atleast one solution is said to be **consistent**.
- A system of linear equations having no solutions is said to be **inconsistent**.

IMPORTANT FORMULA TO REMEMBER

- Co – factor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}
- For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$
  $AA^{-1} = A^{-1}A = I_n$
- If A is non – Singular then
  (i) $|A^{-1}| = \dfrac{1}{|A|}$
  (ii) $(A^T)^{-1} = (A^{-1})^T$
  (iii) $(\lambda A^{-1}) = \dfrac{1}{\lambda} A^{-1}$ where λ is a non – zero scalar.

Reversal law for inverses :

- $(AB)^{-1} = B^{-1}A^{-1}$ where A, B are non – singular matrices of same order.

Law of double inverse :

- If A is non - singular, A^{-1} is also non – singular and $(A^{-1})^{-1} = A$.
- If A is a non - singular square matrix of order n, then
  (i) $(\text{adj } A)^{-1} = \text{adj } (A^{-1}) = \dfrac{1}{|A|} \cdot A$
  (ii) $|\text{adj } A| = |A|^{n-1}$
  (iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$
  (iv) $\text{adj } (\lambda A) = \lambda^{n-1} \text{adj } (A)$ where λ is a non – zero scalar
  (v) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$
  (vi) $(\text{adj } A)^T = \text{adj } (A^T)$
- If a matrix contains at least one non – zero element, then $\rho(a) \geq 1$.
- The rank of identity matrix I_n is n.
  If A is an $m \times n$ matrix then $\rho(A) \leq \min\{m, n\}$.
- A square matrix A of order n is invertible if and only if $\rho(A) = n$.
- Transforming a non-singular matrix A to the form I_n, by applying row operations is called Gauss – Jordan method.

Matrix – Inversion method :

- The solution for $AX = B$ is $X = A^{-1}B$ where A and B are square matrices of same order and non – singular

Cramer's Rule :

- If $\Delta = 0$, Cramer's rule cannot be applied $x_1 = \dfrac{\Delta_1}{\Delta}, x_2 = \dfrac{\Delta_2}{\Delta}, x_3 = \dfrac{\Delta_3}{\Delta}$

Gaussian Elimination method :

Transform the augmented matrix of the system of linear equations into row – echelon form and then solve by back substitution method.

Rouches capelli Theorem :

A system of equations $AX = B$ is consistent if and if $\rho(A) = \rho([A|B])$

(i) If $\rho(A) = \rho([A|B]) = n$, the number of unknowns, then the system is consistent and has a unique solution.

(ii) If $\rho(A) = \rho([A|B]) = n - k,\ k \neq 0$ then the system is consistent and has infinitely many solutions.

(iii) If $\rho(A) \neq \rho([A|B])$, then the system is inconsistent and has no solution.

Homogeneous system of linear equations :

(i) If $\rho(A) = \rho([A|B]) = n$, then the system has a unique solution which is the trivial solution for trivial solution, $|A| \neq 0$

(ii) If $\rho(A) = \rho([A|O]) < n$, the system has a non – trivial solution.

For non – trivial solution, $|A| = 0$.

✦ $A^{-1} = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj } A$ ✦ $A = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj (adj A)}$

EXERCISE 1.1

1. Find the adjoint of the following :

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ **(iii)** $\dfrac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Sol. **(i)** Let A $= \begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

adj A $= \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$

[Interchange the elements in the leading diagonal and change the sign of the elements in off diagonal]

(ii) Let A $= \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$$\text{adj A} = \begin{pmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj A} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) Let A $= \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $\lambda = \dfrac{1}{3}$

Since adj $(\lambda A) = \lambda^{n-1}(\text{adj } A)$

we get adj $\left(\dfrac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\right) = \left(\dfrac{1}{3}\right)^2$

adj $\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

∴ Required adjoint matrix

$$= \frac{1}{9}\begin{bmatrix} +\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} (2+4) & -(-4-2) & +(4-1) \\ -(4+2) & +(4-1) & +(-4-2) \\ +(4-1) & -(4+2) & +(2+4) \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{3}{9}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

[Taking 3 common from each entry]

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse (if it exists) of the following :

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Sol. **(i)** Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Since A is non – singular, A^{-1} exists

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{Now, adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

[Inter change the entries in leading diagonal and change the sign of elements in the off diagonal]

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

(ii) Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Expanding along R_1,

$$|A| = 5\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$

$$= 5\,(25 - 1) - 1\,(5 - 1) + 1\,(1 - 5)$$

$$= 5\,(24) - 1\,(4) + 1\,(-4)$$

$$= 120 - 4 - 4 = 120 - 8 = 112 \neq 0$$

Since A is non singular, A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(25-1) & -(5-1) & +(1-5) \\ -(5-1) & +(25-1) & -(5-1) \\ +(1-5) & -(5-1) & +(25-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

Taking 4 common from every entry we get,

$$\text{adj } A = 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{112}\cdot 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$= \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Expanding along R_1 we get,

$$|A| = 2\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$

$$= 2\,(8 - 7) - 3\,(6 - 3) + 1\,(21 - 12)$$

$$= 2\,(1) - 3(3) + 1\,(9)$$

$$= 2 - \cancel{9} + \cancel{9} = 2 \neq 0$$

Since A is a non-singular matrix, A^{-1} exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$ Show that $[F(\alpha)]^{-1} = F(-\alpha)$ **[Hy - 2019]**

Sol. Given that $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$.

Expanding along R_1 we get,

$$|F(\alpha)| = \cos\alpha \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} - 0 + \sin\alpha \begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix}$$

$= \cos\alpha\,(\cos - 0) + \sin\alpha\,(0 + \sin\alpha)$

$= \cos^2 + \sin^2\alpha = 1 \neq 0$

Since $F(\alpha)$ is a non-singular matrix, $[F(\alpha)]^{-1}$ exists.

Now, adj $(F(\alpha))$ =

$$\begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ \sin\alpha & \cos\alpha \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & \sin\alpha \\ 0 & \cos\alpha \end{vmatrix} & +\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} & -\begin{vmatrix} \cos\alpha & 0 \\ -\sin\alpha & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & \sin\alpha \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(\cos\alpha - 0) & -(0) & +(0 + \sin\alpha) \\ -(0) & +(\cos^2\alpha + \sin^2\alpha) & -(0) \\ +(0 - \sin\alpha) & -(0) & +(\cos - 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & 0 & +\sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\therefore F(\alpha)^{-1} = \frac{1}{|F(\alpha)|} \text{adj } (F(\alpha))$$

$$[F(\alpha)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ +\sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad ...(1)$$

$$\text{Now, } F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad ... (2)$$

[$\because$ $\cos\alpha$ is an even function, $\cos(-\alpha) = \cos\alpha$ and $\sin\alpha$ is an odd function, $\sin(-\alpha) = -\sin\alpha$]

From (1) and (2)

$[F(\alpha)]^{-1} = F(-\alpha)$

Hence proved.

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}.

Sol. Given $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$\therefore A^2 - 3A - 7I_2$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9+0 \\ -3+3+0 & 1+6-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$$

Hence proved.

$\therefore A^2 - 3A - 7I_2 = 0$

Post – multiplying by A^{-1} we get,

$A^2.A^{-1} - 3AA^{-1} - 7I_2 A^{-1} = 0.A^{-1}$

$\Rightarrow A(AA^{-1}) - 3(AA^{-1}) - 7(A^{-1}) = 0$

[$\because I_2 A^{-1} = A^{-1}$ and $(0)A^{-1} = 0$]

$\Rightarrow AI - 3I - 7A^{-1} = 0$ [$\because AA^{-1} = I$]

$\Rightarrow AI - 3I = 7A^{-1}$

$\Rightarrow A^{-1} = \frac{1}{7}[A - 3I]$ [$\therefore AI = A$]

$$\Rightarrow A^{-1} = \frac{1}{7}\left[\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$.

Sol. To prove $A^{-1} = A^T$

$AA^{-1} = AA^T$

It is enough to prove $AA^T = I$

$$AA^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32-28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \frac{81}{81}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore $A^{-1} = A^T$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A| I_2$. **[Sep. - 2020]**

Sol. Given $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$\text{adj } A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

[Interchange the elements in the leading diagonal and change the sign of the elements in the off diagonal]

$|A| = 24 - 20 = 4$

$$\therefore A(\text{adj } A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad ...(1)$$

$$(\text{adj } A)(A) = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad ...(2)$$

$$|A| I_2 = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad ...(3)$$

From (1), (2) and (3), it is proved that $A(\text{adj } A) = (\text{adj } A)A = |A| I_2$ is verified.

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. Given $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$$

$|AB| = -77 + 90 = 13 \neq 0 \Rightarrow (AB)^{-1}$ exists

$|A| = 15 - 14 = 1 \neq 0 \Rightarrow A^{-1}$ exists

$|B| = -2 + 15 = 13 \neq 0 \Rightarrow B^{-1}$ exists

$$(AB)^{-1} = \frac{1}{|AB|}\text{adj}(AB) = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix} \quad ...(1)$$

$$B^{-1} = \frac{1}{|B|}(\text{adj } B) = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$\therefore B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix} \quad ...(2)$$

From (1) and (2) it is prove that $(AB)^{-1} = B^{-1}A^{-1}$.

8. If $\text{adj}(A) = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

Sol. Given $\text{adj } A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

We know that $A = \pm\frac{1}{\sqrt{|\text{adj} A|}}.\text{adj}(\text{adj } A) \quad ...(1)$

$$|\text{adj } A| = 2\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix}$$

[Expanded along R_1]

$= 2(24 - 0) + 4(-6 - 14) + 2(0 + 24)$

$= 2(24) + 4(-20) + 2(24) = 48 - 80 + 48$

$= 96 - 80 = 16$

Now, adj (adj A)

$$= \begin{bmatrix} +\begin{vmatrix}12 & -7\\0 & 2\end{vmatrix} & -\begin{vmatrix}-3 & -7\\-2 & 2\end{vmatrix} & +\begin{vmatrix}-3 & 12\\-2 & 0\end{vmatrix} \\ -\begin{vmatrix}-4 & 2\\0 & 2\end{vmatrix} & +\begin{vmatrix}2 & 2\\-2 & 2\end{vmatrix} & -\begin{vmatrix}2 & -4\\-2 & 0\end{vmatrix} \\ +\begin{vmatrix}-4 & 2\\12 & -7\end{vmatrix} & -\begin{vmatrix}2 & 2\\-3 & -7\end{vmatrix} & +\begin{vmatrix}2 & -4\\-3 & 12\end{vmatrix}\end{bmatrix}^T$$

$$= \begin{bmatrix} +(24-0) & -(-6-14) & +(0+24) \\ -(-8-0) & +(4+4) & -(0-8) \\ +(28-24) & -(-14+6) & +(24-12)\end{bmatrix}^T$$

$$= \begin{bmatrix}24 & 20 & 24\\8 & 8 & 8\\4 & 8 & 12\end{bmatrix}^T = \begin{bmatrix}24 & 8 & 4\\20 & 8 & 8\\24 & 8 & 12\end{bmatrix}$$

$$= 4\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix} \quad ...(3)$$

Substituting (2) and (3) in (1) we get,

$$A = \pm\frac{1}{\sqrt{16}}.4\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix}$$

$$A = \pm\frac{4}{4}\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix} = \pm\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix}$$

9. If adj(A) = $\begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix}$, find A^{-1}. **[PTA-6]**

Sol. Given adj (A) $= \begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix}$

We know that $A^{-1} = \pm\frac{1}{\sqrt{|adj A|}}(adj A)$ ...(1)

$$|adj A| = 0 + 2\begin{vmatrix}6 & -6\\-3 & 6\end{vmatrix} + 0$$

[Expanded along R_1]

$$= 2(36-18) = 2(18) = 36$$

$$\therefore A^{-1} = \pm\frac{1}{\sqrt{36}}\begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix} = \pm\frac{1}{6}\begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix}$$

10. Find adj (adj (A)) if adj A = $\begin{bmatrix}1 & 0 & 1\\0 & 2 & 0\\-1 & 0 & 1\end{bmatrix}$.

Sol. Given adj A $= \begin{bmatrix}1 & 0 & 1\\0 & 2 & 0\\-1 & 0 & 1\end{bmatrix}$

Now adj(adj A) $= \begin{bmatrix} +\begin{vmatrix}2 & 0\\0 & 1\end{vmatrix} & -\begin{vmatrix}0 & 0\\-1 & 1\end{vmatrix} & +\begin{vmatrix}0 & 2\\-1 & 0\end{vmatrix} \\ -\begin{vmatrix}0 & 1\\0 & 1\end{vmatrix} & +\begin{vmatrix}1 & 1\\-1 & 1\end{vmatrix} & -\begin{vmatrix}1 & 0\\-1 & 0\end{vmatrix} \\ +\begin{vmatrix}0 & 1\\2 & 0\end{vmatrix} & -\begin{vmatrix}1 & 1\\0 & 0\end{vmatrix} & +\begin{vmatrix}1 & 0\\0 & 2\end{vmatrix}\end{bmatrix}^T$

$$= \begin{bmatrix} +(2-0) & -(0) & +(0+2) \\ -(0) & +(1+1) & -(0) \\ +(0-2) & -(0) & +(2-0)\end{bmatrix}^T = \begin{bmatrix}2 & 0 & 2\\0 & 2 & 0\\-2 & 0 & 2\end{bmatrix}^T$$

$$adj\ (adj\ A) = \begin{bmatrix}2 & 0 & -2\\0 & 2 & 0\\2 & 0 & 2\end{bmatrix}$$

11. A = $\begin{bmatrix}1 & \tan x\\-\tan x & 1\end{bmatrix}$, show that

$$\mathbf{A^T A^{-1}} = \begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}.$$

Sol. $A = \begin{bmatrix}1 & \tan x\\-\tan x & 1\end{bmatrix}$

$$|A| = \begin{vmatrix}1 & \tan x\\-\tan x & 1\end{vmatrix} = 1 + \tan^2 x = \sec^2 x$$

$$adj\ A = \begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

We know, $A^{-1} = \frac{1}{|A|} Adj\ A$

$$= \frac{1}{\sec^2 x}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix} = \cos^2 x\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

$$= \cos^2 x\begin{bmatrix}1 & -\frac{\sin x}{\cos x}\\ \frac{\sin x}{\cos x} & 1\end{bmatrix}$$

$$A^{-1} = \begin{bmatrix}\cos^2 x & -\cos x \sin x\\ \cos x \sin x & \cos^2 x\end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix} \begin{bmatrix} \cos^2 x & -\cos x \sin x \\ \cos x \sin x & \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -2\sin x \cos x \\ 2\sin x \cos x & \cos^2 x - \sin^2 x \end{bmatrix} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

12. Find the matrix A for which

$$\mathbf{A} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}.$$

Sol. Given $A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Let $B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$\therefore AB = C$

Post multiply by B^{-1} we get

$$A(BB^{-1}) = CB^{-1}$$

$$\Rightarrow A = CB^{-1} \quad [\because BB^{-1} = I]$$

$$|B| = \begin{vmatrix} 5 & 3 \\ -1 & -2 \end{vmatrix} = -10 + 3 = -7 \neq 0$$

$\therefore B^{-1}$ exists

$$B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{-1}{7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$A = CB^{-1}$$

$$= \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \cdot \left(\frac{-1}{7}\right) \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= 7\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \left(\frac{-1}{7}\right) \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= -\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= -\begin{bmatrix} -4+1 & -6+5 \\ -2+1 & -3+5 \end{bmatrix} = -\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that AXB = C.

Sol. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Also, $AXB = C$

Pre-multiply by A^{-1} we get,

$$(A^{-1}A)XB = A^{-1}C$$

$$\Rightarrow XB = A^{-1}.C. \quad [\because A^{-1}A = I]$$

Post Multiply by B^{-1} we get

$$(XB)B^{-1} = (A^{-1}C)B^{-1}$$

$$\Rightarrow X = (A^{-1}C)B^{-1}$$

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 = 2 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 + 2 = 5 \neq 0$$

$$\therefore B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1}C = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore X = (A^{-1}C).B^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1-1 & 2+3 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \frac{1}{5}(5)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$.

Sol. Given $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = 0 - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -1(0-1) + 1(1-0) = 1 + 1 = 2$$

$|A| = 2 \neq 0$, hence A^{-1} exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix}0&1\\1&0\end{vmatrix} & -\begin{vmatrix}1&1\\1&0\end{vmatrix} & +\begin{vmatrix}1&0\\1&1\end{vmatrix} \\ -\begin{vmatrix}1&1\\1&0\end{vmatrix} & +\begin{vmatrix}0&1\\1&0\end{vmatrix} & -\begin{vmatrix}0&1\\1&1\end{vmatrix} \\ +\begin{vmatrix}1&1\\0&1\end{vmatrix} & -\begin{vmatrix}0&1\\1&1\end{vmatrix} & +\begin{vmatrix}0&1\\1&0\end{vmatrix}\end{bmatrix}^T$$

$$= \begin{bmatrix}(0-1) & -(0-1) & +(1-0)\\ -(0-1) & +(0-1) & -(0-1)\\ +(1-0) & -(0-1) & +(0-1)\end{bmatrix}^T$$

$$= \begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}^T = \begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix} \quad ...(1)$$

$$\text{Now } A^2 = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$$

$$= \begin{bmatrix}0+1+1 & 0+0+1 & 0+1+0\\ 0+0+1 & 1+0+1 & 1+0+0\\ 0+1+0 & 1+0+0 & 1+1+0\end{bmatrix} = \begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix} - 3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}2-3&1-0&1-0\\1-0&2-3&1-0\\1-0&1-0&2-3\end{bmatrix}$$

$$= \frac{1}{2}(A^2-3I) = \frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix} \quad ...(2)$$

From (1) and (2), it is proved that $A^{-1} = \frac{1}{2}[A^2 - 3I]$

15. Decrypt the received encoded message [2 –3] [20 4] with the encryption matrix $\begin{bmatrix}-1&-1\\2&1\end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1 – 26 to the letters A – Z respectively, and the number 0 to a blank space.

Sol. Let the encryption matrix be $A = \begin{bmatrix}-1&-1\\2&1\end{bmatrix}$

$$|A| = -1 + 2 = 1 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{bmatrix}1&1\\-2&-1\end{bmatrix} = \begin{bmatrix}1&1\\-2&-1\end{bmatrix}$$

Hence the decryption matrix is $\begin{bmatrix}1&1\\-2&-1\end{bmatrix}$

Coded row matrix	Decoding matrix	Decoded row matrix
[2 – 3]	$\begin{bmatrix}1&1\\-2&-1\end{bmatrix}$	= [2 + 6 2 + 3] = [8 5]
[20 4]	$\begin{bmatrix}1&1\\-2&-1\end{bmatrix}$	= [20 – 8 20 – 4] = [12 16]

So, the sequence of decoded row matrices is [8 5], [12 16]
Now the 8th English alphabet is H.
5th English alphabet is E.
12th English alphabet is L.
and the 16th English alphabet is P.
Thus the receiver reads the message as "HELP".

EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix}2&-4\\-1&2\end{bmatrix}$ (ii) $\begin{bmatrix}-1&3\\4&-7\\3&-4\end{bmatrix}$

(iii) $\begin{bmatrix}1&-2&-1&0\\3&-6&-3&1\end{bmatrix}$ **[PTA - 5]**

(iv) $\begin{bmatrix}1&-2&3\\2&4&-6\\5&1&-1\end{bmatrix}$ (v) $\begin{bmatrix}0&1&2&1\\0&2&4&3\\8&1&0&2\end{bmatrix}$

Sol. (i) Let $A = \begin{bmatrix}2&-4\\-1&2\end{bmatrix}$

A is a matrix of order 2 × 2

$\therefore \rho(A) \leq \min(2, 2) = 2$

The highest order of minor of A is 2

It is $\begin{vmatrix}2&-4\\-1&2\end{vmatrix} = 4 - 4 = 0$

So, $\rho(A) < 2$

Next consider the minor of order 1 $|2| = 2 \neq 0$

$\therefore \rho(A) = 1$

(ii) Let $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

A is a matrix of order 3 × 2

$\therefore \rho(A) \leq \; = \; 2$

We find that there is a second order minor,

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$

$\therefore \rho(A) = 2.$

(iii) Let $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

A is a matrix of order (2 × 4)

$\therefore \rho(A) \leq \min(2, 4) = 2$

The highest order of minor of A is 2

It is $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6 + 6 = 0$

Also, $\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1 + 0 = -1 \neq 0.$

$\therefore \rho(A) = 2.$

(iv) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

A is a matrix of order 3 × 3

$\therefore \rho(A) \leq \min(3, 3) = 3$

The highest order of minor of A is 3.

It is $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$ [Expanded along R_1]

$= 1(-4 + 6) + 2(-2 + 30) + 3(2 - 20)$

$= 1(2) + 2(28) + 3(-18)$

$= 2 + 56 - 54 = 58 - 54 = 4 \neq 0$

$\therefore \rho(A) = 3.$

(v) Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

A is a matrix of order 3 × 4

$\therefore \rho(A) \leq \min(3, 4) = 3$

The highest order of minor of A is 3

It is $\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 8 & 1 & 0 \end{vmatrix} = 0 + 0 - 8(4 - 4) = 0$

[Expanded along C_1]

Also, $\begin{vmatrix} 0 & 2 & 1 \\ 0 & 4 & 3 \\ 8 & 0 & 2 \end{vmatrix} = 0 + 0 - 8\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$

[Expanded along C_1]

$= -8(6 - 4) = -8(2) = -16 \neq 0$

$\therefore \rho(A) = 3$

2. Find the rank of the following matrices by row reduction method :

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ **[PTA-1]**

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Sol. **(i)** Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 5R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The last equivalent matrix is in row echelon form it has two non-zero rows

$\therefore \rho(A) = 2.$

(ii) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 3R_1,\; R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 \div 4} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 3R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$$\xrightarrow{R_3\to 7R_3-R_2}\begin{bmatrix}1&2&-1\\0&-7&5\\0&0&2\\0&0&-1\end{bmatrix}\xrightarrow{R_4\to 2R_4+R_3}\begin{bmatrix}1&2&-1\\0&-7&5\\0&0&2\\0&0&0\end{bmatrix}$$

The last equivalent matrix is in row echelon form it has two non-zero rows

$\therefore$ ρ(A) = 3

(iii) Let $A=\begin{bmatrix}3&-8&5&2\\2&-5&1&4\\-1&2&3&-2\end{bmatrix}$

$$A\xrightarrow{R_3\leftrightarrow R_1}\begin{bmatrix}-1&2&3&-2\\2&-5&1&4\\3&-8&5&2\end{bmatrix}$$

$$\xrightarrow[R_3\to R_3+3R_1]{R_2\to R_2+2R_1}\begin{bmatrix}-1&2&3&-2\\0&-1&7&0\\0&-2&14&-4\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3\div 2}\begin{bmatrix}-1&2&3&-2\\0&-1&7&0\\0&-1&7&-2\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-R_2}\begin{bmatrix}-1&2&3&-2\\0&-1&7&0\\0&0&0&-2\end{bmatrix}$$

The last equivalent matrix is in row-echelon form. It has three non-zero rows.

$\therefore$ ρ(A) = 3

3. Find the inverse of each of the following by Gauss – Jordan method :

(i) $\begin{bmatrix}2&-1\\5&-2\end{bmatrix}$ **[Govt. MQP-2019]**

(ii) $\begin{bmatrix}1&-1&0\\1&0&-1\\6&-2&-3\end{bmatrix}$ **(iii)** $\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}$

Sol. **(i)** Let A $=\begin{bmatrix}2&-1\\5&-2\end{bmatrix}$

Applying Gauss – Jordan method, we get

$$[A|I_2]=\left[\begin{array}{cc|cc}2&-1&1&0\\5&-2&0&1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1\div 2}\left[\begin{array}{cc|cc}1&-\frac{1}{2}&\frac{1}{2}&0\\5&-2&0&1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2-5R_1}\left[\begin{array}{cc|cc}1&-\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&-\frac{5}{2}&1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2\times 2}\left[\begin{array}{cc|cc}1&-\frac{1}{2}&\frac{1}{2}&0\\0&1&-5&2\end{array}\right]$$

$$\xrightarrow{R_1\to R_1+\frac{1}{2}R_2}\left[\begin{array}{cc|cc}1&0&-2&1\\0&1&-5&2\end{array}\right]$$

$\therefore$ We get $A^{-1}=\begin{bmatrix}-2&1\\-5&2\end{bmatrix}$

(ii) Let $A=\begin{bmatrix}1&-1&0\\1&0&-1\\6&-2&-3\end{bmatrix}$

Applying Gauss – Jordan method, we get

$$[A|I_3]=\left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\1&0&-1&0&1&0\\6&-2&-3&0&0&1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2-R_1}\left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\0&1&-1&-1&1&0\\6&-2&-3&0&0&1\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-6R_1}\left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\0&1&-1&-1&1&0\\0&4&-3&-6&0&1\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-4R_2}\left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\0&1&-1&-1&1&0\\0&0&1&-2&-4&1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1+R_2}\left[\begin{array}{ccc|ccc}1&0&-1&0&1&0\\0&1&-1&-1&1&0\\0&0&1&-2&-4&1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1+R_3}\left[\begin{array}{ccc|ccc}1&0&0&-2&-3&1\\0&1&-1&-1&1&0\\0&0&1&-2&-4&1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2+R_3}\left[\begin{array}{ccc|ccc}1&0&0&-2&-3&1\\0&1&0&-3&-3&1\\0&0&1&-2&-4&1\end{array}\right]$$

So, we get $A^{-1}=\begin{bmatrix}-2&-3&1\\-3&-3&1\\-2&-4&1\end{bmatrix}$

(iii) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Applying Gauss Jordan method, we get

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 8R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 \times (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

So, we get $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

(i) $2x + 5y = -2, x + 2y = -3$

(ii) $2x - y = 8, 3x + 2y = -2$ **[PTA -3]**

(iii) $2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1$

(iv) $x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13.$

Sol. **(i)** $2x + 5y = -2, x + 2y = -3$

The matrix form of the system is

$$= \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$\Rightarrow$ AX = B where

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} -2 \\ -3 \end{pmatrix},$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ $X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1 \neq 0.$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 15 \\ -2 + 6 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$\therefore x = -11, y = 4.$

(ii) $2x - y = 8, 3x + 2y = -2$

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ AX = B where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$,

$$B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ $X = A^{-1}B.$

Now, $|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} \frac{14}{7} \\ \frac{-28}{7} \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$\therefore x = 2, y = -4$

Hence, the solution set is $\{2, -4\}$

(iii) $\mathbf{2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1.}$

The matrix form of the system is

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$\Rightarrow$ AX = B where $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$

$\Rightarrow X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$$

[Expanded along R_1]

$= 2(-1+1) - 3(-1-3) - 1(-1-3)$

$= 0 - 3(-4) - 1(-4) = 12 + 4 = 16.$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-1+1) & -(-1-3) & +(-1-3) \\ -(-3-1) & +(-2+3) & -(-2-9) \\ +(3+1) & -(2+1) & +(2-3) \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\therefore x = 2, y = 3, z = 4$

(iv) $\mathbf{x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0,}$ $\mathbf{5x + 2y + 2z = 13}$

The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

AX = B where $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$

$\Rightarrow X = A^{-1}B$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix}$$

$= 1(-8-10) - 1(12-25) + 1(12+20)$

$= 1(-18) - 1(-13) + 1(32) = -18 + 13 + 32 = 27$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} & +\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-8-10) & -(12-25) & +(12+20) \\ -(2-2) & +(2-5) & -(2-5) \\ +(5+4) & -(5-6) & +(-4-6) \end{bmatrix}^T$$

$$= \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}^T = \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{27}\begin{bmatrix}-18 & 0 & 9\\ 13 & -3 & 1\\ 32 & 3 & -10\end{bmatrix}\begin{bmatrix}2\\31\\13\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{27}\begin{bmatrix}-36+0+117\\ 26-93+13\\ 64+93-130\end{bmatrix} = \frac{1}{27}\begin{bmatrix}81\\-54\\27\end{bmatrix} = \begin{bmatrix}3\\-2\\1\end{bmatrix}$$

$\therefore x = 3, y = -2, z = 1$

$\therefore$ Solution set is $\{3, -2, 1\}$

2. If $A = \begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}$ **and** $B = \begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$**, find the products AB and BA and hence solve the system of equations** $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2$.

Sol. Given $A = \begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}$, $B = \begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$

$$AB = \begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}-5+3+6 & -5+2+3 & -10+1+9\\ 7+3-10 & 7+2-5 & 14+1-15\\ 1-3+2 & 1-2+1 & 2-1+3\end{bmatrix}$$

$$= \begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix} = 4.\, I_3$$

$$BA = \begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-5+7+2 & 1+1-2 & 3-5+2\\ -15+14+1 & 3+2-1 & 9-10+1\\ -10+7+3 & 2+1-3 & 6-5+3\end{bmatrix}$$

$$= \begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix} = 4.\, I_3.$$

So, we get $AB = BA = 4.\, I_3$

$$\Rightarrow \left(\frac{1}{4}A\right)B = B\left(\frac{1}{4}A\right) = I$$

$$\Rightarrow B^{-1} = \frac{1}{4}A$$

Writing the given set of equations in matrix form we get,

$$\begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\7\\2\end{bmatrix}$$

$$\Rightarrow B\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\7\\2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = B^{-1}\begin{bmatrix}1\\7\\2\end{bmatrix} = \left[\frac{1}{4}A\right]\begin{bmatrix}1\\7\\2\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\7\\2\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-5+7+6\\ 7+7-10\\ 1-7+2\end{bmatrix} = \frac{1}{4}\begin{bmatrix}8\\4\\-4\end{bmatrix} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$$

$\therefore x = 2, y = 1, z = -1$

Hence, the solution set is $\{2, 1, -1\}$.

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹ 19,800 per month at the end of the first month after 3 years of service and ₹ 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

Sol. Let the man's starting salary be ₹ x and his annual increment be ₹ y.

By the given data $x + 3y = 19800$ and $x + 9y = 23,400$.

The matrix form of the given system of equations is

$$\begin{bmatrix}1 & 3\\ 1 & 9\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}19800\\23400\end{bmatrix}$$

$\Rightarrow AX = B$ where $A = \begin{bmatrix}1 & 3\\ 1 & 9\end{bmatrix}$ and $B = \begin{bmatrix}19800\\23400\end{bmatrix}$

$$\Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix}1 & 3\\ 1 & 9\end{vmatrix} = 9 - 3 = 6 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}\,A = \frac{1}{6}\begin{bmatrix}9 & -3\\ -1 & 1\end{bmatrix}$$

$$\therefore X = A^{-1}B$$

$$= \frac{1}{6}\begin{bmatrix}9 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}19800\\23400\end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 178200 - 70200 \\ -19800 + 23400 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 108000 \\ 3600 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$$

$\therefore$ $x = 18000, y = 600$.

Hence the man's starting salary is ₹ 18000 and his annual increment is ₹ 600.

4. Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Sol. Let the time by one man alone be x days and one woman alone be y days

$\therefore$ By the given data,

$$\frac{4}{x} + \frac{4}{y} = \frac{1}{3} \text{ and } \frac{2}{x} + \frac{5}{y} = \frac{1}{4}$$

$$\text{put } \frac{1}{x} = s \text{ and } \frac{1}{y} = t$$

$$\therefore 4s + 4t = \frac{1}{3} \text{ and } 2s + 5t = \frac{1}{4}$$

The matrix form of the system of equation is

$$\begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \Rightarrow AX = B \text{ where}$$

$$A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$X = A^{-1} B$$

$$\text{Now } |A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix} = 20 - 8 = 12 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$$\therefore X = A^{-1} B = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{5}{3} - 1 \\ \frac{-2}{3} + 1 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \times \frac{1}{12} \\ \frac{1}{3} \times \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$$

$$\therefore s = \frac{1}{18} \Rightarrow \frac{1}{x} = \frac{1}{18} \Rightarrow x = 18$$

$$t = \frac{1}{36} \Rightarrow \frac{1}{y} = \frac{1}{36} \Rightarrow y = 36.$$

Hence, the time taken by 1 man alone is 18 days and the time taken by 1 woman alone is 36 days.

5. The prices of three commodities A, B and C are ₹ x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P,Q and R earn ₹ 15,000, ₹ 1,000 and ₹ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Sol. Let the prices per unit for the commodities A, B and C be ₹ x, ₹ y and ₹ z.

By the given data,

$$2x - 4y + 5z = 15000$$
$$3x + y - 2z = 1000$$
$$-x + 3y + z = 4000$$

The matrix form of the system of equations is

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$AX = B \text{ where } A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\text{and } B = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} B$$

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= 2(1 + 6) + 4(3 - 2) + 5(9 + 1)$$

$$= 2(7) + 4(1) + 5(10) = 14 + 4 + 50 = 68.$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix}1 & -2\\3 & 1\end{vmatrix} & -\begin{vmatrix}3 & -2\\-1 & 1\end{vmatrix} & +\begin{vmatrix}3 & 1\\-1 & 3\end{vmatrix} \\ -\begin{vmatrix}-4 & 5\\3 & 1\end{vmatrix} & +\begin{vmatrix}2 & 5\\-1 & 1\end{vmatrix} & -\begin{vmatrix}2 & -4\\-1 & 3\end{vmatrix} \\ +\begin{vmatrix}-4 & 5\\1 & -2\end{vmatrix} & -\begin{vmatrix}2 & 5\\3 & -2\end{vmatrix} & +\begin{vmatrix}2 & -4\\3 & 1\end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(1+6) & -(3-2) & +(9+1) \\ -(-4-15) & +(2+5) & -(6-4) \\ +(8-5) & -(-4-15) & +(2+12) \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 105000+19000+12000 \\ -15000+7000+76000 \\ 150000-2000+56000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$\therefore x = 2000, y = 1000, z = 3000.$

Hence the prices per unit of the commodities A, B and C are ₹ 2000, ₹ 1000 and ₹ 3000 respectively.

EXERCISE 1.4

1. Solve the following systems of linear equation by Cramer's rule:

(i) $5x - 2y + 16 = 0, \; x + 3y - 7 = 0$

(ii) $\frac{3}{x} + 2y = 12, \; \frac{2}{x} + 3y = 13$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9,$ $4x + 3y + 2z = 25$ **[Hy - 2019]**

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \; \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,$ $\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Sol. (i)

$$\text{Given } \Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\Delta_1 = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_2 = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-34}{17} = -2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{51}{17} = 3$$

$x = -2, y = 3.$

(ii) Let $\frac{1}{x} = z$

$$\therefore 3z + 2y = 12, \; 2z + 3y = 13$$

$$\therefore \Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$$

$$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

$$\therefore z = \frac{\Delta_1}{\Delta} = \frac{10}{5} = 2 \Rightarrow \frac{1}{x} = 2 \Rightarrow x = \frac{1}{2}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$$

$$\therefore x = \frac{1}{2}, y = 3$$

(iii)

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3(-2-6) - 3(4-8) - 1(6+4)$$

$$= 3(-8) - 3(-4) - 1(10)$$

$$= -24 + 12 - 10 = -22$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$$= 11\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 1\begin{vmatrix} 9 & -1 \\ 25 & 3 \end{vmatrix}$$

$= 11(-2-6) - 3(18-50) - 1(27+25)$

$= 11(-8) - 3(-32) - 1(52)$

$= -88 + 96 - 52 = -44$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 11\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix}$$

$= 3(18-50) - 11(4-8) - 1(50-36)$

$= 3(-32) - 11(-4) - 1(14)$

$= -96 + 44 - 14 = -66$

$$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 9 \\ 3 & 25 \end{vmatrix} - 3\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix} + 11\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$= 3(-25-27) - 3(50-36) + 11(6+4)$

$= 3(-52) - 3(14) + 11(10)$

$= -156 - 42 + 110 = -88$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-44}{-22} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-66}{-22} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-88}{-22} = 4$$

$\therefore x = 2, y = 3, z = 4.$

(iv) Put $\frac{1}{x} = X, \frac{1}{y} = Y, \frac{1}{z} = Z$

We get $3X - 4Y - 2Z = 1$, $X + 2Y + Z = 2$, $2X - 5Y - 4Z = -1$

$$\therefore \Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$= 3(-8+5) + 4(-4-2) - 2(-5-4)$

$= 3(-3) + 4(-6) - 2(-9)$

$= -9 - 24 + 18 = -15$

$$\Delta_x = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & -5 \end{vmatrix}$$

$= 1(-8+5) + 4(-8+1) - 2(-10+2)$

$= 1(-3) + 4(-7) - 2(-8)$

$= -3 - 28 + 16 = -15$

$$\Delta_y = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$= 3(-8+1) - 1(-4-2) - 2(-1-4)$

$= 3(-7) - 1(-6) - 2(-5)$

$= -21 + 6 + 10 = -5$

$$\Delta_z = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$= 3(-2+10) + 4(-1-4) + 1(-5-4)$

$= 3(8) + 4(-5) + 1(-9)$

$= 24 - 20 - 9 = -5$

$$\therefore X = \frac{\Delta_x}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

$$Y = \frac{\Delta_y}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3$$

$$Z = \frac{\Delta_z}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{z} = \frac{1}{3} \Rightarrow z = 3$$

$\therefore x = 1, y = 3, z = 3.$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem). **[Qy - 2019]**

Sol. Let x represent the number of question with correct answer and y represent the number of questions with wrong answers.

By the given data, $x + y = 100$ and ... (1)

$$1.x - \frac{1}{4}y = 80$$

Multiplying by 4 we get,

$$4x - y = 320 \quad \text{... (2)}$$

From (1) and (2)

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1 - 4 = -5$$

$$\Delta_1 = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100 - 320 = -420$$

$$\Delta_2 = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix} = 320 - 400 = -80$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-420}{-5} = +84$$

$$\text{and } y = \frac{\Delta_2}{\Delta} = \frac{-80}{-5} = 16$$

Hence, the number of questions with correct answer is 84.

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

Sol. Let the amount of 50% acid be x litres and the amount of 25% acid be y litres

By the given data, $x + y = 10$ ... (1)

$$\text{and } x\left(\frac{50}{100}\right) + y\left(\frac{25}{100}\right) = 10\left(\frac{40}{100}\right)$$

$$\Rightarrow 50x + 25y = 400 \Rightarrow 2x + y = 16 \quad \text{... (2)}$$

$$\Delta = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = -1$$

$$\Delta_x = \begin{vmatrix} 10 & 1 \\ 16 & 1 \end{vmatrix} = 10 - 16 = -6$$

$$\Rightarrow \Delta_y = \begin{vmatrix} 1 & 10 \\ 2 & 16 \end{vmatrix} = 16 - 20 - 4$$

$$x = \frac{\Delta_x}{\Delta_y} = \frac{-6}{-1} = 6$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-4}{-1} = 4$$

i.e., 6 litres of 50% acid and 4 litres of 25% acid solution to be mixed to get 10 litres of 40% of acid solution.

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

Sol. Let the pump A can fill the tank in x minutes, and the pump B can fill the tank in y minutes

In 1 minute A can fill $\frac{1}{x}$ units and in 1 minute B can fill $\frac{1}{y}$ units

$$\therefore \frac{1}{x} + \frac{1}{y} = 10$$

$$\text{and } \frac{1}{x} - \frac{1}{y} = 30$$

$$\text{Put } \frac{1}{x} = a \text{ and } \frac{1}{y} = b$$

$$\Rightarrow a + b = \frac{1}{10} \quad \text{... (1)}$$

$$\text{and } a - b = \frac{1}{30} \quad \text{... (2)}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$\Delta_1 = \begin{vmatrix} \frac{1}{10} & 1 \\ \frac{1}{30} & -1 \end{vmatrix} = \frac{-1}{10} - \frac{1}{30}$$

$$= \frac{-3-1}{30} = \frac{-4}{30} = \frac{-2}{15}$$

$$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{10} \\ 1 & \frac{1}{30} \end{vmatrix} = \frac{1}{30} - \frac{1}{10} = \frac{1-3}{30}$$

$$= \frac{-2}{30} = \frac{-1}{15}$$

$$\therefore a = \frac{\Delta_1}{\Delta} = \frac{\frac{-2}{15}}{-2} = \frac{1}{15} \Rightarrow \frac{1}{x} = \frac{1}{15} \Rightarrow x = 15$$

$$b = \frac{\Delta_2}{\Delta} = \frac{\frac{-1}{15}}{-2} = \frac{1}{30} \Rightarrow \frac{1}{y} = \frac{1}{30} \Rightarrow y = 30$$

Hence the pump A can fill the tank in 15 minutes and the pump B can fill the tank in 30 minutes.

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹ 150. The cost of the two dosai, two idlies and four vadais is ₹ 200. The cost of five dosai, four idlies and two vadais is ₹ 250. The family has ₹ 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

Sol. Let the cost of one dosa be ₹ x

The cost of one idli be ₹ y

and the cost of one vadai be ₹ z

By the given data,

$$2x + 3y + 2z = 150$$
$$2x + 2y + 4z = 200$$
$$5x + 4y + 2z = 250$$

$$\therefore \Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$= 2(4 - 16) - 3(4 - 20) + 2(8 - 10)$

$= 2(-12) - 3(-16) + 2(-2)$

$= -24 + 48 - 4 = 20$

$$\Delta_1 = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

Taking 50 common from C_1 and 2 common from C_3 we get,

$$=100\begin{vmatrix} 3 & 3 & 1 \\ 4 & 2 & 2 \\ 5 & 4 & 1 \end{vmatrix} = 100\left[3\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix}\right]$$

$= 100[3(2 - 8) - 3(4 - 10) + 1(16 - 10)]$

$= 100[3(-6) - 3(-6) + 6]$

$= 100[-18 + 18 + 6] = 600.$

$$\Delta_2 = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix} = 100\begin{vmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 5 & 5 & 1 \end{vmatrix}$$

$$= 100\left[2\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix}\right]$$

$= 100[2(4 - 10) - 3(2 - 10) + 1(10 - 20)]$

$= 100[2(-6) - 3(-8) + 1(-10)]$

$= 100[-12 + 24 - 10] = 100 [2] = 200.$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix} = 50\begin{vmatrix} 2 & 3 & 3 \\ 2 & 2 & 4 \\ 5 & 4 & 5 \end{vmatrix}$$

$$= 50\left[2\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}\right]$$

$= 50 [2(10 - 16) - 3(10 - 20) + 3(8 - 10)]$

$= 50[2(-6) - 3(-10) + 3(-2)]$

$= 50 [-12 + 30 - 6] = 50 [12] = 600.$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{600}{20} = 30$$

$$y = \frac{\Delta_2}{\Delta} = \frac{200}{20} = 10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{600}{20} = 30.$$

Hence, the price of one dosa be ₹ 30, one idli be ₹10 and the price of 1 vadai be ₹ 30.

Also the cost of 3 dosa, six idlies and six vadai is

$= 3x + 6y + 6z = 3(30) + 6(10) + 6(30)$

$= 90 + 60 + 180 =$ ₹ 330

Since the family had ₹ 350 in hand, they will be able to manage to pay the bill.

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method :

(i) $2x - 2y + 3z = 2, x + 2y - z = 3, 3x - y + 2z = 1.$

(ii) $2x + 4y + 6z = 22, 3x + 8y + 5z = 27,$ $-x + y + 2z = 2$

Sol. (i) Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -1 & 0 & -4 \end{array}\right] \xrightarrow{R_3 \to 6R_3 - 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right]$$

Writing the equivalent equations from the row-echelon matrix, we get,

$$x + 2y - z = 3 \quad \text{... (1)}$$
$$-6y + 5z = -4 \quad \text{... (2)}$$
$$-5z = -20 \Rightarrow z = \frac{-20}{-5} = 4.$$

Substituting $z = 4$ in 2 we get,

$$-6y + 5(4) = -4$$
$$\Rightarrow -6y + 20 = -4 \Rightarrow -4 - 20 = -24$$
$$\Rightarrow y = \frac{-24}{-6} = 4.$$

Substituting $y = z = 4$ in (1) we get

$$x + 2(4) - 4 = 3$$
$$\Rightarrow x + 8 - 4 = 3$$
$$\Rightarrow x + 4 = 3$$
$$\Rightarrow x = 3 - 4 = -1.$$

$\therefore x = -1, y = 4, z = 4.$

(ii) $2x + 4y + 6z = 22$, $3x + 8y + 5z = 27$, $-x + y + 2z = 2$

Reducing the augmented matrix to an equivalent row echelon form by using elementary row operations, we get

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & 8 & 5 & 27 \\ 2 & 4 & 6 & 22 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + 3R_1} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 11 & 11 & 33 \\ 0 & 6 & 10 & 26 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 2]{R_2 \to R_2 \div 11} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 \div 2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Writing the equivalent equations from the row echelon matrix we get,

$$-x + y + 2z = 2 \quad \text{...(1)}$$
$$y + z = 3 \quad \text{...(2)}$$
$$z = 2 \quad \text{...(3)}$$

Substituting (3) in (2) we get, $y + 2 = 3$

$$\Rightarrow y = 3 - 2 = 1$$

Substituting $y = 1$ and $z = 2$ in (1) we get,

$$-x + 1 + 2(2) = 2 \quad \Rightarrow -x + 1 + 4 = 2$$
$$\Rightarrow -x + 5 = 2 \quad \Rightarrow -x = 2 - 5$$
$$\Rightarrow -x = -3 \quad \Rightarrow x = 3$$

$\therefore x = 3, y = 1, z = 2.$

2. If $ax^2 + bx + c$ is divided by $x + 3$, $x - 5$, and $x - 1$, the remainders are 21, 61 and 9 respectively. Find a, b and c. (Use Gaussian elimination method.) **[PTA -3]**

Sol. Let $P(x) = ax^2 + bx + c$

Given $P(-3) = 21$

[$\because P(x) \div x + 3$, the remainder is 21]

$$\Rightarrow a(-3)^2 + b(-3) + c = 21$$
$$\Rightarrow 9a - 3b + c = 21 \quad \text{...(1)}$$

Also, $P(5) = 61$

$$\Rightarrow a(5)^2 + b(5) + c = 61$$

[using remainder theorem]

$$\Rightarrow 25a + 5b + c = 61 \quad \text{...(2)}$$

and $P(1) = 9$

$$\Rightarrow a(1)^2 + b(1) + c = 9$$
$$\Rightarrow a + b + c = 9 \quad \text{...(3)}$$

Reducing the augment matrix to an equivalent row-echelon form using elementary row operations, we get

$$\left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 25R_1]{R_3 \to R_3 - 9R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 4]{R_2 \to R_2 \div 4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & -3 & -2 & -15 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{3}{5}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & \frac{8}{5} & \frac{48}{5} \end{array}\right]$$

$$\xrightarrow{R_3 \to 5R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & 8 & 48 \end{array}\right]$$

Writing the equivalent equations from the row-echelon matrix we get,

$$a + b + c = 9 \quad ...(1)$$
$$-5b - 6c = -41 \quad ...(2)$$
$$8c = 48$$
$$\Rightarrow c = \frac{48}{8} = 6$$

Substituting $c = 6$ in (2) we get,

$$\Rightarrow -5b - 6(6) = -41$$
$$\Rightarrow -5b = 36 - 41$$
$$\Rightarrow -5b = -41 + 36 = -5$$
$$\Rightarrow b = \frac{-5}{-5} = 1$$

Substituting $b = 1$, $c = 6$ in (1) we get,

$$a + 1 + 6 = 9$$
$$\Rightarrow a + 7 = 9$$
$$\Rightarrow a = 9 - 7$$
$$\Rightarrow a = 2$$

$\therefore a = 2, b = 1$ and $c = 6$

3. An amount of ₹ 65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is ₹ 4,800. The income from the third bond is ₹ 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Sol. Let the price of bond invested in 6%, 8% and 9% rates be let ₹ x, ₹ y and ₹ z respectively

$\therefore$ By the given data, $x + y + z = 65{,}000$ ...(1)

$$\frac{6 \times x \times 1}{100} + \frac{8 \times y \times 1}{100} + \frac{9 \times z \times 1}{100} = 4{,}800$$

$$\left[\because \text{Interest} = \frac{PNR}{100}\right]$$

$$\Rightarrow \frac{6x}{100} + \frac{8y}{100} + \frac{9z}{100} = 4{,}800$$
$$\Rightarrow 6x + 8y + 9z = 4{,}80{,}000 \quad ...(2)$$

$$\text{Also, } \frac{9z}{100} = 600 + \frac{8y}{100}$$
$$\Rightarrow \frac{-8y}{100} + \frac{9z}{100} = 600$$
$$\Rightarrow -8y + 9z = 60{,}000 \quad ...(3)$$

Reducing the augmented matrix to an equivalent row-echelon form by using elementary row operation, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 65{,}000 \\ 6 & 8 & 9 & 4{,}80{,}000 \\ 0 & -8 & 9 & 60{,}000 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 6R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65{,}000 \\ 0 & 2 & 3 & 90{,}000 \\ 0 & -8 & 9 & 60{,}000 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65{,}000 \\ 0 & 2 & 3 & 90{,}000 \\ 0 & 0 & 21 & 4{,}20{,}000 \end{array}\right]$$

Writing the equivalent from the row echelon matrix we get,

$$x + y + z = 65{,}000 \quad ...(1)$$
$$2y + 3z = 90{,}000 \quad ...(2)$$
$$21z = 4{,}20{,}000$$
$$\Rightarrow z = \frac{4{,}20{,}000}{21} = 20{,}000$$

Substituting z = 20,000 in (2),

$$2y + 3(20{,}000) = 90{,}000$$
$$\Rightarrow 2y + 60{,}000 = 90{,}000$$
$$\Rightarrow 2y = 90{,}000 - 60{,}000$$
$$= 30{,}000$$
$$\Rightarrow y = \frac{30{,}000}{2} = 15{,}000$$

Substituting $y = 15{,}000$ and $z = 20{,}000$ in (1) we get,

$$x + 15{,}000 + 20{,}000 = 65{,}000$$
$$\Rightarrow x + 35{,}000 = 65{,}000$$
$$\Rightarrow x = 65{,}000 - 35{,}000$$
$$\Rightarrow x = 30{,}000$$

Thus the price of 6% bond is ₹ 30,000 the price of 8% bond is ₹ 15,000 and the price of 9% bond is ₹ 20,000.

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points (–6, 8),(–2 – 12) and (3,8). He wants to meet his friend at P(7,60). Will he meet his friend? (Use Gaussian elimination method.)

Sol. Given $y = ax^2 + bx + c$ ...(1)

(–6, 8) lies on (1)

$$\Rightarrow 8 = a(-6)^2 + b(-6) + c$$
$$\Rightarrow 8 = 36a - 6b + c \quad ...(2)$$

(–2, –12) lies on (1)

$\Rightarrow \quad -12 = a(-2)^2 + b(-2) + c$

$\Rightarrow \quad -12 = 4a - 2b + c \quad ...(3)$

Also (3, 8) lies on (1)

$\Rightarrow \quad 8 = a(3)^2 + b(3) + c$

$\Rightarrow \quad 8 = 9a + 3b + c \quad ...(4)$

Reducing the augment matrix to an equivalent row-echelon form by using elementary row operations, we get,

$$\left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right] \xrightarrow[R_3 \to 4R_3 - R_1]{R_2 \to 9R_2 - R_1} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 3]{R_2 \to R_2 \div 4} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right]$$

Writing the equivalent equation from the row echelon matrix, we get $36a - 6b + c = 8 \quad ...(1)$

$-3b + 2c = -29 \quad ...(2)$

$5c = -50$

$\Rightarrow \quad c = \frac{-50}{5} = -10$

Substituting $c = -10$ in (2) we get,

$-3b + 2(-10) = -29$

$\Rightarrow \quad -3b - 20 = -29$

$\Rightarrow \quad -3b = -29 + 20$

$\Rightarrow \quad -3b = -9$

$\Rightarrow \quad b = \frac{-9}{-3} = 3$

Substituting $b = 3$ and $c = -10$ in (1) we get,

$36a - 6(3) - 10 = 8$

$\Rightarrow \quad 36a - 18 - 10 = 8$

$\Rightarrow \quad 36a - 28 = 8$

$\Rightarrow \quad 36a = 8 + 28 = 36$

$\Rightarrow \quad a = \frac{36}{36} = 1$

$\therefore \; a = 1, b = 3, c = -10$

Hence the path of the boy is

$y = 1(x^2) + 3(x) - 10$

$\Rightarrow \quad y = x^2 + 3x - 10$

Since his friend is at P(7, 60),

$60 = (7)^2 + 3(7) - 10$

$\Rightarrow \quad 60 = 49 + 21 - 10$

$\Rightarrow \quad 60 = 70 - 10 = 60$

$\Rightarrow \quad 60 = 60$

Since (7, 60) satisfies his path , he can meet his friend who is at P(7, 60)

EXERCISE 1.6

1. Test for consistency and if possible, solve the following systems of equations by rank method.

(i) $x - y + 2z = 2, 2x + y + 4z = 7, 4x - y + z = 4$

(ii) $3x + y + z = 2, x - 3y + 2z = 1, 7x - y + 4z = 5.$

(iii) $2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$ **[Sep. - 2020]**

(iv) $2x - y + z = 2, 6x - 3y + 3z = 6, 4x - 2y + 2z = 4.$ **[PTA - 5;Hy - 2019]**

Sol. **(i)** $x - y + 2z = 2, 2x + y + 4z = 7, 4x - y + z = 4$

The matrix form of the system is AX = B

where $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$

Applying elementary row operations on the augment matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{array}\right]$$

Here $\rho(A) = 3$ and $\rho[A|B] = 3$

$\therefore \; \rho(A) = \rho[A|B] = 3$ = number of unknowns

Hence the system is consistent with unique solution.

Writing the equivalent equations from the row-echelon matrix, we get

$$x - y + 2z = 2 \quad ...(1)$$
$$3y = 3 \Rightarrow y = 1 \quad ...(2)$$
$$-7z = -7 \Rightarrow z = \frac{-7}{-7} = 1 \quad ...(3)$$

Solutions $y = 1$ and $z = 1$ in (1) we get,

$$x - 1 + 2(1) = 2$$
$$\Rightarrow x - 1 + 2 = 2$$
$$\Rightarrow x + 1 = 2$$
$$\Rightarrow x = 2 - 1 = 1$$
$$\Rightarrow x = 1$$

$\therefore x = 1, y = 1, z = 1$

(ii) $3x + y + z = 2, x - 3y + 2z = 1, 7x - y + 4z = 5.$

Then matrix form of the system is AX = B

where $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

Applying elementary row operations on the augment matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$, and $\rho[A|B] = 2$ [since there only two non-zero rows]. So, $\rho(A) = \rho[A|B] = 2 < 3$, the given system is consistent with one parameter family of solutions. So, put $z = t$, $x \in \mathbb{R}$. Writing the equivalent equations from the row echelon matrix we get,

$$x - 3y + 2z = 1 \quad ...(1)$$
$$10y - 5z = -1 \quad ...(2)$$
$$z = t \quad ...(3)$$

(2) becomes $10y - 5t = -1$

$$\Rightarrow 10y = 5t - 1$$
$$\Rightarrow y = \frac{1}{10}[5t - 1]$$

Also, from (1), $x - \frac{3}{10}[5t - 1] + 2t = 1$

$$\Rightarrow x = \frac{3}{10}[5t - 1] - 2t + 1 = \frac{15t - 3 - 20t + 10}{10}$$
$$\Rightarrow x = \frac{1}{10}[-5t + 7]$$

Hence the solution set is

$x = \frac{1}{10}(7 - 5t), y = \frac{1}{10}(5t - 1)$ and $z = t$ where $t \in \mathbb{R}$.

(iii) $2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$

The matrix form of the given system is AX = B where

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

Applying elementary row operations on the augmented matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

Here $\rho(A) = 2$ [$\because$ There are 2 non-Zero rows]

and $\rho[A|B] = 3$ [$\because$ There are 3 non –zero rows]

Here, $\rho(A) \neq \rho[A|B]$

Hence, the given system is inconsistent and has no solution.

(iv) $2x - y + z = 2, 6x - 3y + 3z = 6, 4x - 2y + 2z = 4.$

The matrix form of the given system is AX = B where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

Applying elementary row operations on the augment matrix[A|B], we get,

$$[A|B]=\left[\begin{array}{ccc|c}2 & -1 & 1 & 2\\ 6 & -3 & 3 & 6\\ 4 & -2 & 2 & 4\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-3R_1}\left[\begin{array}{ccc|c}2 & -1 & 1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

Here $\rho(A) = 1$[$\because$ only one non zero row]

and $\rho[A|B] = 1$[$\because$ only one-zero row]

$\therefore$ $\rho(A) = \rho[A|B] = 1 < 3$, the given system is consistent and has two parameter family of solutions.

So, $z = t$ and $y = s$ where s, $t \in \mathbb{R}$.

Writing the equivalent equations from the row-echelon matrix, we get

$$2x - y + z = 2 \quad ...(1)$$
$$y = s \quad ...(2)$$
$$z = t \quad ...(3)$$

Substituting (2) and (3) in (1) we get,

$$2x - s + t = 2$$
$$\Rightarrow \quad 2x = s - t + 2$$
$$\Rightarrow \quad x = \frac{1}{2}[s - t + 2]$$

$\therefore$ Solution set is $x = \frac{1}{2}(s - t + 2), y = s, z = t$ where $s,\ t \in \mathbb{R}$

2. Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2$, $x - 2y + kz = 1$ have **[Qy - 2019]**

(i) no solution

(ii) unique solution

(iii) infinitely many solution

Sol. $kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + kz = 1$

The matrix form of the system is AX = B where

$$A = \begin{bmatrix}k & -2 & 1\\ 1 & -2k & 1\\ 1 & -2 & k\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, B = \begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$

Applying elementary row operation on the augment matrix [A|B] we get,

$$[A/B] = \left[\begin{array}{ccc|c}k & -2 & 1 & 1\\ 1 & -2k & 1 & -2\\ 1 & -2 & k & 1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_3}\left[\begin{array}{ccc|c}1 & -2 & k & 1\\ 1 & -2k & 1 & -2\\ k & -2 & 1 & 1\end{array}\right]$$

$$\xrightarrow[R_3\to R_3-kR_1]{R_2\to R_2-R_1}\left[\begin{array}{ccc|c}1 & -2 & k & 1\\ 0 & -2k+2 & 1-k & -3\\ 0 & -2+2k & 1-k^2 & 1-k\end{array}\right]$$

$$\xrightarrow{R_3\to R_3+R_2}\begin{bmatrix}1 & -2 & k & 1\\ 0 & -2k+2 & 1-k & -3\\ 0 & 0 & 1-k^2+1-k & -k-2\end{bmatrix}$$

$$\to\begin{bmatrix}1 & -2 & k & 1\\ 0 & -2k+2 & 1-k & -3\\ 0 & 0 & -k^2-k+2 & -k-2\end{bmatrix}$$

$$\to\begin{bmatrix}1 & -2 & k & 1\\ 0 & -2k+2 & 1-k & -3\\ 0 & 0 & (k+2)(1-k) & -k-2\end{bmatrix} \quad ...(1)$$

Case (i) when $k = 1$

$$[A|B] \to \begin{bmatrix}1 & -2 & 1 & 1\\ 0 & 0 & 0 & -3\\ 0 & 0 & 0 & -3\end{bmatrix}\xrightarrow{R_3\to R_3-R_2}\begin{bmatrix}1 & -2 & 1 & 1\\ 0 & 0 & 0 & -3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Here $\rho(A) = 1$ and $\rho[A|B] = 2$

So, $\rho(A) \neq \rho[A|B] \Rightarrow$ The system has no solution.

Case (ii) When $k \neq 1, k \neq -2$

$$[A|B] \to \left[\begin{array}{ccc|c}1 & -2 & k & 1\\ 0 & -2k+2 & 1-k & -3\\ 0 & 0 & \text{not zero} & \text{not zero}\end{array}\right]$$

$\Rightarrow \rho(A) = 3$ and $\rho[A|B] = 3$

so, $\rho(A) = \rho[A|B] = 3$ = the number of unknowns

Hence, the system has unique solution.

Case (iii) when $k = -2$

$$\rho[A|B] \to \begin{bmatrix}1 & -2 & -2 & 1\\ 1 & 6 & 3 & -3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Here $\rho(A) = 2$ and $\rho[A|B] = 2$

$\therefore \rho(A) = \rho[A|B] = 2 < 3$, the number of unknowns so the system is consistent with infinitely many solutions.

3. Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9$, $7x + 3y - 5z = 8$, $2x + 3y + \lambda z = \mu$, have

(i) no solution

(ii) a unique solution

(iii) an infinite number of solutions.

Sol. $2x + 3y + 5z = 9, 7x + 3y - 5z = 8, 2x + 3y + \lambda z = \mu$

The matrix form of the system is AX = B where

$$A = \begin{bmatrix}2 & 3 & 5\\ 7 & 3 & -5\\ 2 & 3 & \lambda\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, B = \begin{bmatrix}9\\ 8\\ \mu\end{bmatrix}$$

Applying elementary row operations on the augmented matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 7 & 3 & -5 & 8 \\ 2 & 3 & 5 & 9 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_2 \to R_2 - \frac{2}{7}R_1} \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & \frac{15}{7} & \frac{45}{7} & \frac{45}{7} \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 \times 7} \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

Case (i) When $\lambda = 5$

$$[A|B] = \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & 0 & \mu-9 \end{array}\right]$$

Here $\rho(A) = 2$ and $\rho[A|B] = 3$

So, $\rho(A) \neq \rho[A|B]$

Hence the system is inconsistent and has no solution

Case (ii) When $\lambda \neq 5$, $\mu \neq 9$

$$[A|B] = \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & \text{not zero} & \text{not zero} \end{array}\right]$$

Here $\rho(A) = 3$ and $\rho[A|B] = 3$

$\therefore$ $\rho(A) = \rho[A|B] = 3$ = number of unknowns

Hence, the system is consistent with unique solution

Case (iii) When $\lambda = 5$ and $\mu = 9$

$$[A|B] = \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$, $\rho[A|B] = 2$

$\therefore$ $\rho(A) = [A|B] = 2 <$ number of unknowns

$\therefore$ The system is consistent and has infinite number of solutions.

EXERCISE 1.7

1. Solve the following system of homogeneous equations.

(i) $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$, $5x + 9y + 23z = 0$

(ii) $2x + 3y - z = 0$, $x - y - 2z = 0$, $3x + y + 3z = 0$.

Sol. **(i)** $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$, $5x + 9y + 23z = 0$

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 4 & -3 & -2 & 0 \\ 5 & 9 & 23 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 17 & 34 & 0 \end{array}\right] \begin{array}{l} R_2 = 3R_2 - 4R_1 \\ R_3 = 3R_3 - 5R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 = R_3 + R_2$$

So, $\rho(AB) = \rho(A) = 2$ and $n = 3$.

Hence, the system has a one parameter family of solutions. Writing the equations using the echelon form, we get

$3x + 2y + 7z = 0$ and $-17y - 34z = 0$

To solve the equations let $z = t$,

$$\begin{aligned} \text{then } -17y - 34t &= 0 \\ -17y &= 34t \\ y &= \frac{34t}{17} \\ y &= -2t \\ \text{Substituting } z &= t \text{ and } y = -2t \text{ in} \\ 3x + 2y + 7z &= 0 \\ \text{We get } 3x + 2(-2t) + 7t &= 0 \\ 3x - 4t + 7t &= 0 \\ 3x + 3t &= 0 \\ 3x &= -3t \\ x &= -\frac{3t}{3} \\ x &= -t, \end{aligned}$$

So the solution is

$x = -t$, $y = -2t$ and $z = t$ where $t \in \mathbb{R}$

(ii) $2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0.$

Here the number of equations is equal to the number of unknowns. Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 9 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 33 & 0 \end{bmatrix} R_3 = 5R_2 - 4R_2$$

So, $\rho(AB) = \rho(A) = 3$ and $n = 3$

Hence, the system has a unique solution.

Since $x = 0, y = 0, z = 0$, is always a solution of the homogeneous system, the only solution is the trivial solution

$x = 0, y = 0, z = 0$

2. Determine the values of λ for which the following system of equations

$x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x + y + 2z = 0$ has

(i) a unique solution

(ii) a non-trivial solution.

Sol. **$x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x + y + 2z = 0$**

Reducing the augmented matrix to row – echelon form we get,

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda - 12 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{array}\right]$$

Case (i) when $\lambda \neq 8$

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \text{not zero} & 0 \end{array}\right]$$

Here $\rho(A) = 3, \rho([A/O]) = 3$

$\therefore \rho(A) = \rho([A/0]) = 3 =$ the number of unknowns

$\therefore$ The given system is consistent and has unique solution.

Case (ii) when $\lambda = 8$

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2, \rho([A|0]) = 2$

$\therefore \rho(A) = \rho([A|0]) = 2 < 3$, the number of unknowns,

$\therefore$ The system is consistent and has non-trivial solutions.

3. By using Gaussian elimination method, balance the chemical reaction equation:

$C_2H_6 + O_2 \rightarrow H_2O + CO_2$ **[Govt. MQP-2019]**

Sol. Given $C_2H_6 + O_2 \rightarrow H_2O + CO_2$

We have to find positive integers x_1, x_2, x_3 and x_4 such that

$x_1 C_2H_6 + x_2 O_2 \rightarrow x_3 H_2O + x_4 CO_2$ ...(1)

The number of carbon atoms on the LHS of (1) should be equal to the number of carbon atoms on the RHS of (1).

$\therefore 2x_1 = 1x_4$

$\Rightarrow 2x_1 - x_4 = 0$ ...(2)

Considering hydrogen atoms we get,

$6x_1 = 2x_3$

$\Rightarrow 6x_1 - 2x_3 = 0$

$\Rightarrow 3x_1 - x_3 = 0$ ...(3)

Also, considering oxygen atoms we get,

$2x_2 = 1x_3 + 2x_4$

$\Rightarrow 2x_2 - x_3 - 2x_4 = 0$ ...(4)

Equations (2), (3) and (4) form a homogeneous system of linear equations in 4 unknowns

$\therefore$ The augmented matrix $[A|0]$ is

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$$

By Gaussian elimination method, we get,

$$\xrightarrow{R_1 \to R_1 \div 2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & -1 & \frac{3}{2} & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & -1 & \frac{3}{2} & 0 \\ 0 & 2 & 0 & \frac{-7}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 2 & 0 & \frac{-7}{2} & 0 \\ 0 & 0 & -1 & \frac{3}{2} & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to 2R_2 \\ R_2 \to 2R_3}]{R_1 \to 2R_1} \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & -7 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{array}\right]$$

Here $\rho(A) = \rho([A|B]) = 3 < 4$, the number of unknowns

$\therefore$ The system is consistent with one parameter family of solutions, so let $x_4 = t$

Writing the equations, from the row-echelon form we get (from equation (2))

$$2x_1 - x_4 = 0$$
$$\Rightarrow \quad 2x_1 = x_4$$
$$\Rightarrow \quad 2x = t \quad \Rightarrow x_1 = \frac{t}{2}$$

From equation (3)

$$\Rightarrow \quad 3x_1 - x_3 = 0$$
$$\Rightarrow \quad x_3 = 3x_1 \quad \Rightarrow x_2 = \frac{3t}{2}$$

From equation (4)

$$2x_2 - x_3 - 2x_4 = 0 \Rightarrow 2x_2 - \frac{3t}{2} - 2t = 0$$
$$\Rightarrow \quad 2x_2 - \frac{7t}{4} = 0 \Rightarrow x_2 = \frac{7t}{4}$$

Since x_1, x_2, x_3 and x_4 are positive integers, let us choose $t = 4$

$$\therefore x_1 = \frac{4}{2} = 2$$
$$\therefore x_2 = \frac{7(\not{4})}{\not{4}} = 7$$
$$\therefore x_3 = \frac{3(\not{4})^{2}}{\not{2}} = 6 \text{ and } x_4 = t = 4$$

So, the balanced equation is

$2\,C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$.

EXERCISE 1.8

Choose the Correct or the most suitable answer from the given four alternatives :

1. If |adj (adj A) | = $|A|^9$, then the order of the square matrix A is

(1) 3 (2) 4 (3) 2 (4) 5

[Ans. (2) 4]

Hint : $|\text{adj (adj)}A| = |A|^{(n-1)^2}$

$$\therefore (n-1)^2 = 9 \Rightarrow (n-1)^2 = 3^2$$
$$\Rightarrow \quad n-1 = 3 \Rightarrow n = 4$$

2. If A is a 3 × 3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then BB^T =

(1) A (2) B (3) I_3 (4) B^T

[Ans. (3) I_3]

Hint : $BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$

$$= (A^{-1}A^T)\,(A^T)^T \cdot (A^{-1})^T$$
$$= (A^{-1}A^T)\,A(A^{-1})^T = A^{-1}(A.A^T)(A^{-1})^T$$
$$= (A^{-1}A).\,A^T\,(A^T)^{-1} \quad [\because (A^{-1})^T = (A^T)^{-1}]$$
$$= I \cdot I = I \quad [\because A^T \cdot (A^T)^{-1} = I]$$

3. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, then $\frac{|\text{adj } B|}{|C|}$ =

[Govt. MQP-2019]

(1) $\frac{1}{3}$ (2) $\frac{1}{9}$ (3) $\frac{1}{4}$ (4) 1

[Ans. (2) $\frac{1}{9}$]

Hint : $\frac{|\text{adj}B|}{|C|} = \frac{|\text{adj(adj } A)|}{|3A|} = \frac{|A|^{(n-1)^2}}{|3|^2.|A|}$

$$= \frac{|A|^2}{|B|^2 \cdot |A|} = \frac{|A|}{9.|A|} = \frac{1}{9}$$

4. If $A\begin{bmatrix}1 & -2\\1 & 4\end{bmatrix} = \begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$, then A =

(1) $\begin{bmatrix}1 & -2\\1 & 4\end{bmatrix}$ (2) $\begin{bmatrix}1 & 2\\-1 & 4\end{bmatrix}$

(3) $\begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$ (4) $\begin{bmatrix}4 & -1\\2 & 1\end{bmatrix}$

[Ans. (3) $\begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$]

Hint : $AX = B$

$\Rightarrow A = BX^{-1}$ where

$X = \begin{bmatrix}1 & -2\\1 & 4\end{bmatrix}, B = \begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$

$A = \begin{bmatrix}6 & 0\\0 & 6\end{bmatrix} \cdot \frac{1}{|X|} \cdot \text{adj}(X)$

$= 6\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix} = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$

5. If $A = \begin{bmatrix}7 & 3\\4 & 2\end{bmatrix}$, then $9I_2 - A$ = [PTA - 2]

(1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ (4) $2A^{-1}$

[Ans. (4) $2A^{-1}$]

Hint : $9I - A = \begin{bmatrix}9 & 0\\0 & 9\end{bmatrix} - \begin{bmatrix}7 & 3\\4 & 2\end{bmatrix}$

$= \begin{bmatrix}9-7 & 0-3\\0-4 & 9-2\end{bmatrix} = \begin{bmatrix}2 & -3\\-4 & 7\end{bmatrix} = \text{adj } A$

But $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2}\text{adj } A \Rightarrow \text{adj } A = 2A^{-1}$

6. If $A = \begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}$ and $B = \begin{bmatrix}1 & 4\\2 & 0\end{bmatrix}$ then |adj (AB)| =

(1) –40 (2) –80 (3) –60 (4) –20

[Ans. (2) –80]

Hint : $AB = \begin{bmatrix}2+0 & 8+0\\1+10 & 4+0\end{bmatrix} = \begin{bmatrix}2 & 8\\11 & 4\end{bmatrix}$

$\text{adj}(AB) = \begin{bmatrix}4 & -8\\-11 & 2\end{bmatrix}$

$|\text{adj}(AB)| = 8 - 88 = -80$

7. If $P = \begin{vmatrix}1 & x & 0\\1 & 3 & 0\\2 & 4 & -2\end{vmatrix}$ is the adjoint of 3 × 3 matrix A and |A| = 4, then x is

(1) 15 (2) 12 (3) 14 (4) 11

[Ans. (4) 11]

Hint : $|\text{adj } A| = |A|^{n-1}$

$1\begin{vmatrix}3 & 0\\4 & -2\end{vmatrix} - x\begin{vmatrix}1 & 0\\2 & -2\end{vmatrix} + 0 = 4^{3-1}$

$\Rightarrow -6 - x(-2) = 4^2 \Rightarrow -6 + 2x = 16$

$\Rightarrow 2x = 22 \Rightarrow x = 11$

8. If $A = \begin{bmatrix}3 & 1 & -1\\2 & -2 & 0\\1 & 2 & -1\end{bmatrix}$ and $A^{-1} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$ then the value of a_{23} is [PTA - 5]

(1) 0 (2) –2 (3) –3 (4) –1

[Ans. (4) –1]

Hint : $|A| = 3\begin{vmatrix}-2 & 0\\2 & -1\end{vmatrix} - 1\begin{vmatrix}2 & 0\\1 & -1\end{vmatrix} - 1\begin{vmatrix}2 & -2\\1 & 2\end{vmatrix}$

$= 3(2) - 1(-2) - 1(4+2)$

$= 6 + 2 - 6 = 2$

$a_{23} = \frac{1}{|A|}\text{ co-factor of } a_{32} = \frac{1}{2} \times -\begin{vmatrix}3 & -1\\2 & 0\end{vmatrix}$

$= -\frac{1}{2}(0+2) = \frac{-2}{2} = -1$

9. If A, B and C are invertible matrices of some order, then which one of the following is not true?

(1) adj A = $|A|A^{-1}$

(2) adj (AB) = (adj A) (adj B)

(3) det A^{-1} = $(\det A)^{-1}$

(4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

[Ans. (2) adj (AB) = (adj A) (adj B)]

10. If $(AB)^{-1} = \begin{bmatrix}12 & -17\\-19 & 27\end{bmatrix}$ and $A^{-1} = \begin{bmatrix}1 & -1\\-2 & 3\end{bmatrix}$, then B^{-1} = [Mar. - 2020]

(1) $\begin{bmatrix}2 & -5\\-3 & 8\end{bmatrix}$ (2) $\begin{bmatrix}8 & 5\\3 & 2\end{bmatrix}$

(3) $\begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}$ (4) $\begin{bmatrix}8 & -5\\-3 & 2\end{bmatrix}$

[Ans. (1) $\begin{bmatrix}2 & -5\\-3 & 8\end{bmatrix}$]

Hint : Since $(AB)^{-1} = B^{-1}A^{-1}$, we get,

$$\begin{bmatrix}12 & -17\\-19 & 27\end{bmatrix} = B^{-1}\begin{bmatrix}1 & -1\\-2 & 3\end{bmatrix}$$

$$\text{Let } X = B^{-1}Y$$

$$B^{-1} = XY^{-1} = \begin{bmatrix}12 & -17\\-19 & 27\end{bmatrix}\cdot\frac{1}{|Y|}.(\text{adj } Y)$$

$$= \begin{bmatrix}12 & -17\\-19 & 27\end{bmatrix}\cdot\frac{1}{1}\begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}12 & -17\\-19 & 27\end{bmatrix}\begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}36-34 & 12-17\\-57+54 & -19+27\end{bmatrix} = \begin{bmatrix}2 & -5\\-3 & 8\end{bmatrix}$$

11. If $A^T. A^{-1}$ is symmetric, then A^2 = [Sep. - 2020]

(1) A^{-1} (2) $(A^T)^2$
(3) A^T (4) $(A^{-1})^2$

[Ans : (2) $(A^T)^2$]

Hint :

$$\Rightarrow A^TA^{-1} = (A^TA^{-1})^T$$
$$= (A^{-1})^T(A^T)^T = (A^{-1})^TA$$
$$\Rightarrow A^TA^{-1} = (A^{-1})^T.A$$
$$\Rightarrow A^TA^{-1} = (A^T)^{-1}.A$$

[Premultiplying by A^T and post multiplying by A on both sides, we get]

$$A^T(A^TA^{-1})A = A^T[(A^T)^{-1}.A]A$$
$$\Rightarrow (A^T)^2(A^{-1}A) = A^T(A^T)^{-1}.A^2$$
$$\Rightarrow (A^T)^2(I) = I.A^2$$
$$\Rightarrow (A^T)^2 = A^2$$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix}5 & 3\\-2 & -1\end{bmatrix}$, then $(A^T)^{-1}$ = [PTA -1 ;Hy - 2019]

(1) $\begin{bmatrix}-5 & 3\\2 & 1\end{bmatrix}$ (2) $\begin{bmatrix}5 & 3\\-2 & -1\end{bmatrix}$

(3) $\begin{bmatrix}-1 & -3\\2 & 5\end{bmatrix}$ (4) $\begin{bmatrix}5 & -2\\3 & -1\end{bmatrix}$

[Ans : (4) $\begin{bmatrix}5 & -2\\3 & -1\end{bmatrix}$]

Hint : $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix}5 & -2\\3 & -1\end{bmatrix}$

13. If A = $\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\x & \frac{3}{5}\end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is

(1) $\frac{-4}{5}$ (2) $\frac{-3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

[Ans: (1) $\frac{-4}{5}$]

Hint : Since $A^T = A^{-1}$, $AA^T = A^TA = I$ [∴ they are orthogonal]

$$\Rightarrow \begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\x & \frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{3}{5} & x\\\frac{4}{5} & \frac{3}{5}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\frac{9}{25}+\frac{16}{25} & \frac{3x}{5}+\frac{12}{25}\\\frac{3x}{5}+\frac{12}{25} & x^2+\frac{9}{25}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\Rightarrow \frac{3x}{5}+\frac{12}{25} = 0 \text{ [Equating } a_{12} \text{ both sides]}$$

$$\Rightarrow \frac{3x}{5} = \frac{-12}{25} \Rightarrow x = \frac{-12}{25}\times\frac{5}{3} = \frac{-4}{5}$$

14. If A = $\begin{bmatrix}1 & \tan\frac{\theta}{2}\\-\tan\frac{\theta}{2} & 1\end{bmatrix}$ and AB = I_2, then B = [PTA -3]

(1) $\left(\cos^2\frac{\theta}{2}\right)A$ (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$

(3) $(\cos^2\theta)I$ (4) $\left(\sin^2\frac{\theta}{2}\right)A$

[Ans: (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$]

Hint :

$$B = A^{-1} = \frac{1}{|A|}\text{ adj } A$$

$$= \frac{1}{1+\tan^2\frac{\theta}{2}}\begin{bmatrix}1 & -\tan\frac{\theta}{2}\\\tan\frac{\theta}{2} & 1\end{bmatrix} = \frac{1}{\sec^2\frac{\theta}{2}}.A^T = \left(\cos^2\frac{\theta}{2}\right)A^T$$

15. If A = $\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$ and A (adj A) = $\begin{bmatrix}k & 0\\0 & k\end{bmatrix}$, then k =

(1) 0 (2) sin θ (3) cos θ (4) 1

[Ans: (4) 1]