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PREFACE

The woods are lovely, dark and deep.
But I have promises to keep, and
miles to go before I sleep

\- Robert Frost

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers, and dear Students.

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **SURA'S Business Mathematics and Statistics** for +2 Standard. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises, additional problems and 1 marks as per new model in addition to precise answers for exercise problems.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

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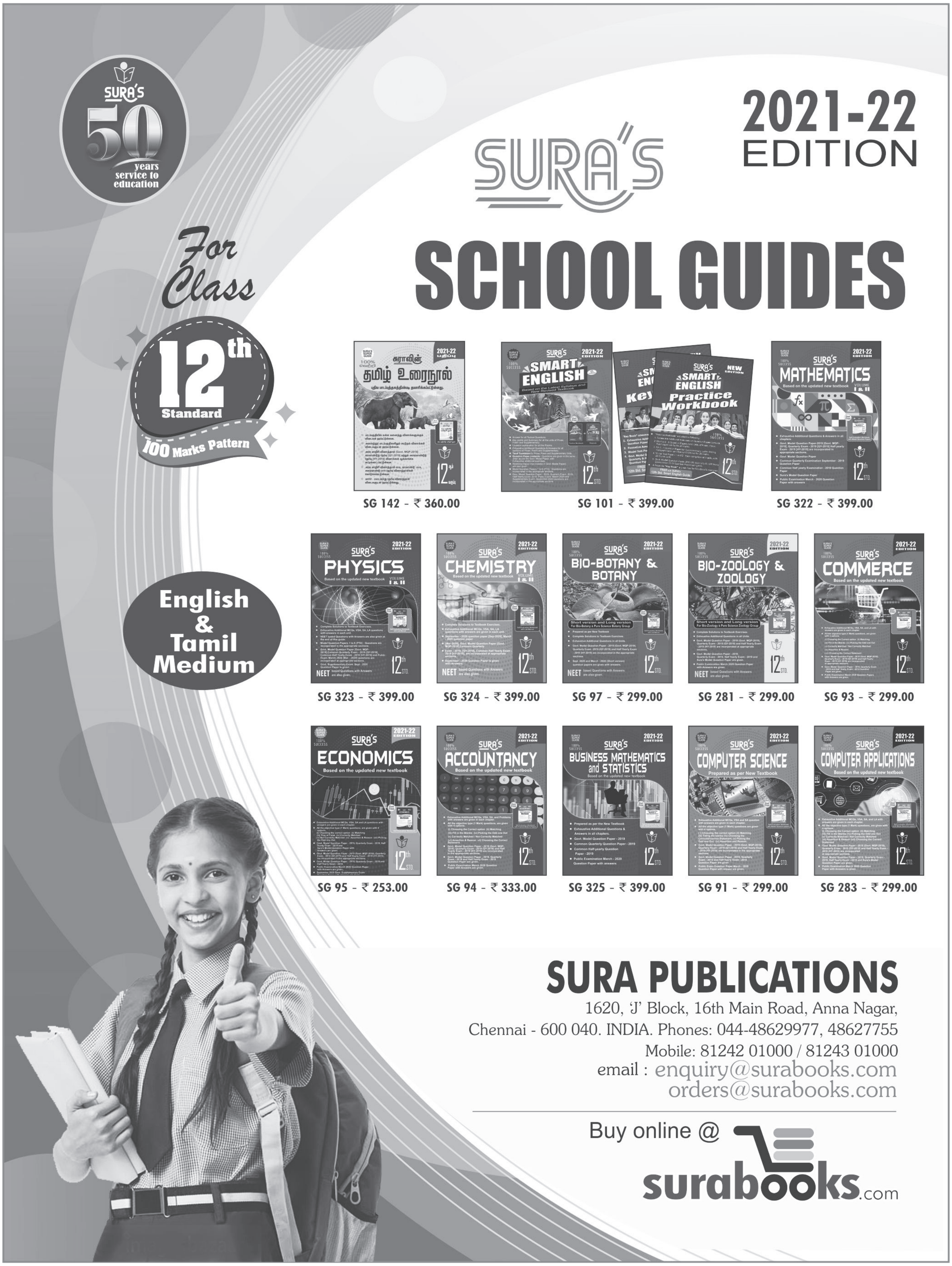
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Chapter 1

APPLICATIONS OF MATRICES AND DETERMINANTS

CHAPTER SNAPSHOT

Rank of a matrix :-

The rank of a matrix A is the order of the largest non-zero minor of A and is denoted by ρ (A).

(i) $\rho(A) \geq 0$.

(ii) If A is a matrix of order $m \times n$, then $\rho(A) \leq \min\{m, n\}$.

(iii) Rank of a zero matrix is 0.

(iv) The rank of a non - singular matrix of order $n \times n$ *is "n"*.

Elementary transformations :

(i) Interchange any two rows (or columns)

$$R_i \leftrightarrow R_j \ (C_i \leftrightarrow C_j)$$

(ii) Multiplication of each element of a row (or column) by any non-zero scalar k.

$$R_i \rightarrow k\,R_i \ (\text{or } C_i \rightarrow k\,C_i)$$

(iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column).

$$R_i \rightarrow R_i + k\,R_j \ (\text{or } C_i \rightarrow C_i + k\,C_i)$$

Equivalent matrices:

Two matrices A and B are said to be equivalent if one is obtained from the another by applying a finite number of elementary transformations.

$A \cong b$

Echelon form :

A matrix A of order $m \times n$ is said to be in echelon form if

(i) Every row of A which has all its entries 0 occurs below every row which has a non - zero entry.

(ii) The number of zeros before the first non - zero element in a row is less than the number of such zeros in the next row.

Transition matrix :

The transition probabilities P_{jk} satisfy $P_{jk} > 0$, $\sum_k P_{jk} = 1$ for all j

FORMULAE TO REMEMBER

1. Linear equations can be written in matrix form AX = B, then the solution is $X = A^{-1}B$, provided $|A| \neq 0$.
2. Consistency of non homogeneous linear equations by rank method.

(i) If $\rho([A,B]) = \rho(A)$, then the equations are consistent.

(ii) If $\rho([A,B]) = \rho(A) = n$, where n is the number of variables then the equations are consistent and have unique solution.

(iii) If $\rho([A,B]) = \rho(A) < n$, then the equations are consistent and have infinitely many solutions.

(iv) If $\rho([A,B]) \neq \rho(A)$, then the equations are inconsistent and has no solution.

3. Solving non-homogeneous linear equations by Cramer's rule.

If $a_1x + b_1y + c_1z = d_1$,
$a_2x + b_2y + c_2z = d_2$,
$a_3x + b_3y + c_3z = d_3$

Then $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, $\Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Then $x = \dfrac{\Delta x}{\Delta}$, $y = \dfrac{\Delta y}{\Delta}$ and $z = \dfrac{\Delta z}{\Delta}$

TEXTUAL QUESTIONS

EXERCISE 1.1

1. Find the rank of each of the following matrices.

(i) $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ **(iv)** $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$ **(vi)** $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

(vii) $\begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$ **(viii)** $\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

Sol : (i) Let A $= \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

Order of A is 2 × 2

$\therefore \rho(A) \leq 2$ [Since minimum of (2, 2) is 2]

Consider the second order minor,

$$\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = 40 - 42 = -2 \neq 0.$$

There is a minor of order 2, which is not zero

$\therefore \rho(A) = 2$

(ii) Let A $= \begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$

Order of A is 2 × 2

$\therefore \rho(A) \leq 2$ [Since minimum of (2, 2) is 2]

Consider the second order minor,

$$\begin{vmatrix} 1 & -1 \\ 3 & -6 \end{vmatrix} = -6-(-3) = -6+3 = -3 \neq 0.$$

There is a minor of order 2, which is not zero

$\therefore \rho(A) = 2.$

(iii) Let A $= \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ **[QY-2019]**

Order of A is 2 × 2 [Since minimum of (2,2) is 2]

Consider the second order minor $\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}$

$= 8 - 8$

$= 0.$

Since the second order minor vanishes, $\rho(A) \neq 2$

Consider a first order minor $|1| \neq 0$

There is a minor of order 1, which is not zero

$\therefore \rho(A) = 1.$

(iv) Let A $= \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$ **[PTA - 1]**

The order of A is 3 × 3

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -5 \\ 2 & -1 & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -8 \\ 0 & -3 & -1 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & -3 & -1 \end{pmatrix}$	$R_2 \to R_2 \div 2$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 11 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

This matrix is in echelon from and number of non-zero rows is 3.

$\therefore \rho(A) = 3.$

(v) Let $A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$

The order of A is 3×3

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$	$R_1 \to R_1 (-1)$
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -4 \\ -2 & 4 & -4 \end{pmatrix}$	$R_2 \to R_2 - 4R_1$
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 + 2R_1$

The matrix is in echelon form and the number of non-zero rows is 2.

$\therefore \rho(A) = 2.$

(vi) Let $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

The order of A is 3×4

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 3 & 6 & 3 & -7 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non- zero rows is 2.

$\therefore \rho(A) = 2.$

(vii) $A = \begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$

The order of A is 3×4

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 4) is 3]

Let us transform the matrix A to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & -5 & -1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 3 & 1 & -5 & -1 \end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 0 & -14 & 16 & -7 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 0 & 0 & 0 & 7 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non-zero matrix is 3.

$\therefore \rho(A) = 3.$

(viii) $A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

The order of A is 3×4

$\therefore \rho(A) \leq$ minimum of $(3, 4) \Rightarrow \rho(A) \leq 3$

Let us transform the matrix A to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ -1 & 2 & 7 & 6 \end{pmatrix}$	$R_2 \to R_2 + 2R_1$
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{pmatrix}$	$R_3 \to R_3 + R_1$
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non- zero rows is 2.

$\therefore \rho(A) = 2.$

2. If A= $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$, then find the rank of AB and the rank of BA.

Sol :

Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2-5 & -2+4-1 & 3-6+1 \\ 2+6+20 & -4-12+4 & 6+18-4 \\ 3+4+15 & -6-8+3 & 9+12-3 \end{pmatrix}$$

$$AB = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$$

Matrix (AB)	Elementary Transformation
$AB = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -6 & -2 \\ -12 & 28 & 20 \\ -11 & 22 & 18 \end{pmatrix}$	$C_1 \leftrightarrow C_2$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ -11 & 22 & 18 \end{pmatrix}$	$R_2 \to R_2 + 12R_1$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & -44 & -4 \end{pmatrix}$	$R_3 \to R_3 + 11R_1$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The matrix is in echelon form and the number of non-zero rows is 2.

$\therefore \rho(AB) = 2.$

Now, $BA = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1-4+9 & 1+6-6 & -1-8+9 \\ -2+8-18 & -2-12+12 & 2+16-18 \\ 5+2-3 & 5-3+2 & -5+4-3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 & 0 \\ -12 & -2 & 0 \\ 4 & 4 & -4 \end{pmatrix}$$

Matrix (BA)	Elementary Transformation
$BA = \begin{pmatrix} 6 & 1 & 0 \\ -12 & -2 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 6 & 0 \\ -2 & -12 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	$C_1 \leftrightarrow C_2$
$\sim \begin{pmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	$R_2 \to R_2 + 2R_1$
$\sim \begin{pmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & -20 & -4 \end{pmatrix}$	$R_3 \to R_3 - 4R_1$

The number of non-zero rows is 2.

$\therefore \rho(BA) = 2.$

3. Solve the following system of equations by rank method $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x - y - z = 0$

Sol : The given equations are $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x - y - z = 0$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix [AB]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{pmatrix}$	$R_3 \to 3R_3 + R_2$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{pmatrix}$	$\Rightarrow P(A) = 3$

Since augmented matrix $[A, B] \sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{pmatrix}$ has three non-zero rows, $\rho([A,B]) = 3$.

That is, $\rho(A) = \rho([A,B]) = 3$ = number of unknowns.

So the given system is consistent and has unique solution.

To find the solution, we rewrite the echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 34 \\ -20 \end{pmatrix}$$

$\Rightarrow \quad x + y + z = 9 \quad \text{... (1)}$

$3y + 5z = 34 \quad \text{...(2)}$

$-4z = -20 \quad \text{...(3)}$

$(3) \Rightarrow \quad -4z = -20$

$z = \dfrac{-20}{-4} = 5$

$(2) \Rightarrow \quad 3y + 5(5) = 34$

$\Rightarrow \quad 3y + 25 = 34 \Rightarrow 3y = 34 - 25$

$\Rightarrow \quad 3y = 9 \quad \Rightarrow y = \dfrac{9}{3}$

$y = 3.$

$(1) \Rightarrow \quad x + 3 + 5 = 9$

$\Rightarrow \quad x + 8 = 9 \quad \Rightarrow x = 9 - 8 \Rightarrow x = 1$

$\therefore x = 1,\ y = 3,\ z = 5$ is the unique solution of the given equations.

4. Show that the equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$ are consistent and solve them by rank method.

Sol : Given non-homogeneous equations are

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

A X = B

Augmented matrix $[A, B] = \begin{pmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{pmatrix}$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	
$\sim\begin{pmatrix} 3 & 26 & 2 & 9 \\ 5 & 3 & 7 & 4 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 5 & 3 & 7 & 4 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_1 \to R_1 \div 3$
$\sim\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-121}{3} & \frac{11}{3} & -11 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_2 \to R_2 - 5R_1$
$\sim\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-121}{3} & \frac{11}{3} & -11 \\ 0 & \frac{-176}{3} & \frac{16}{3} & -16 \end{pmatrix}$	$R_3 \to R_3 - 7R_1$
$\sim\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \end{pmatrix}$	$R_2 \to R_2 \div 11$ $R_3 \to R_3 \div 16$
$\sim\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Here $\rho(A) = \rho(A, B) = 2 <$ Number of unknowns.

$\therefore$ The system is consistent with infinitely many solutions let us rewrite the above echelon form into matrix form.

$$\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} \\ 0 & \frac{-11}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$x + \frac{26}{3}y + \frac{2}{3}z = 3 \quad \text{... (1)}$$

$$\frac{-11}{3}y + \frac{1}{3}z = -1 \quad \text{...(2)}$$

let $z = k$; where $k \in R$

$(2) \Rightarrow \frac{-11}{3}y + \frac{k}{3} = -1$

$\Rightarrow \frac{-11}{\cancel{3}}y = -1 - \frac{k}{3} = \frac{-3-k}{\cancel{3}}$

$\Rightarrow -11y = -3 - k$

$\Rightarrow 11y = 3 + k$

$\Rightarrow y = \frac{1}{11}(3 + k)$

Substituting $y = \frac{1}{11}(3 + k)$ and $z = k$ in (1) we get,

$x + \frac{26}{3}\left(\frac{3+k}{11}\right) + \frac{2}{3}k = 3$

$x = -\frac{26}{3}\left(\frac{3+k}{11}\right) - \frac{2k}{3} + 3$

$= \frac{-78-26k}{33} - \frac{2k}{3} + 3 = \frac{-78-26k-22k+99}{33}$

$= \frac{21-48k}{33} = \frac{3(7-16k)}{33}$

$x = \frac{1}{11}(7 - 16k)$

$\therefore$ Solution set is $\{\frac{1}{11}(7-16k), \frac{1}{11}(3+k), k\}\ k \in R$.

Hence, for different values of k, we get infinitely many solutions.

5. Show that the following system of equations have unique solution: $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$ by rank method. [QY-2019]

Sol : Given non-homogeneous equations are

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

A X = B

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

Clearly the last equivalent matrix is in echelon form and it has three non-zero rows.

$\therefore$ $\rho(A) = 3$ and $\rho([A, B]) = 3$

$\Rightarrow \rho(A) = \rho([A, B]) = 3 =$ Number of unknowns.

$\therefore$ The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x + y + z = 3$ ... (1)

$y + 2z = 1$ ... (2)

$2z = 0$ ... (3)

$(3) \Rightarrow 2z = 0 \Rightarrow z = \frac{0}{2} = 0$

$(2) \Rightarrow y + 2(0) = 1 \Rightarrow y + 0 = 1 \Rightarrow y = 1 - 0 = 1$

$(1) \Rightarrow x + 1 + 0 = 3$

$\Rightarrow x + 1 = 3$

$\Rightarrow x = 3 - 1$

$\Rightarrow x = 2$

$\therefore$ Solution is $\{2, 1, 0\}$

6. For what values of the parameter λ, will the following equations fail to have unique solution: $3x - y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$ by rank method.

Sol : Given non-homogeneous equations are

$$3x - y + \lambda z = 1$$
$$2x + y + z = 2$$
$$x + 2y - \lambda z = -1$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 3 & -1 & \lambda \\ 2 & 1 & 1 \\ 1 & 2 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

A X = B

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 3 & -1 & \lambda & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & -\lambda & -1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 3 & -1 & \lambda & 1 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & -7 & 4\lambda & 4 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -1 & \frac{1+2\lambda}{3} & \frac{4}{3} \\ 0 & -1 & \frac{4\lambda}{7} & \frac{4}{7} \end{pmatrix}$	$R_2 \to R_2 \div 3$ $R_3 \to R_3 \div 7$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -1 & \frac{1+2\lambda}{3} & \frac{4}{3} \\ 0 & 0 & \frac{-7-2\lambda}{21} & \frac{-16}{21} \end{pmatrix}$	$R_3 \to R_3 - R_2$

Since

$$\frac{4\lambda}{7} - \frac{1+2\lambda}{3} = \frac{12\lambda - 7 - 14\lambda}{21} = \frac{-7-2\lambda}{21}$$

$$\text{and } \frac{4}{7} - \frac{4}{3} = \frac{12-28}{21} = \frac{-16}{21}$$

Since the system is fail to have unique solution either it can have infinitely many solution or it may be inconsistent.

$\therefore$ This can happen only when $\dfrac{-7-2\lambda}{21} = 0$.

$$\Rightarrow \quad -7-2\lambda = 0$$
$$\Rightarrow \quad -7 = 2\lambda$$
$$\Rightarrow \quad \lambda = \frac{-7}{2}.$$

7. The price of three commodities X,Y and Z are *x,y* and *z* respectively Mr. Anand Purchases 6 units of Z and sells 2 units of X and 3 units of Y. Mr. Amar Purchases a unit of Y and sells 3 units of X and 2units of Z. Mr. Amit Purchases a unit of X and sells 3 units of Y and a unit of Z. In the process they earn ₹ 5,000/-, ₹ 2,000/- and ₹ 5,500/- respectively Find the prices per unit of three commodities by rank method. **[PTA-5]**

Sol : Given that the price of commodities X, Y and Z are x, y and z respectively.

By the given data,

Transaction	x	y	z	Earning
Mr. Anand	+2	+3	–6	Rs. 5000
Mr. Amar	+3	–1	+2	Rs. 2000
Mr. Amit	–1	+3	+1	Rs. 5500

Here, purchasing is taken as negative symbol and selling is taken as positive symbol.

Thus, the non-homogeneous equations are

$$2x + 3y - 6z = 5000$$
$$3x - y + 2z = 2000$$
$$-x + 3y + z = 5500$$

The matrix equation corresponding to the given system is

$$\underset{A}{\begin{pmatrix} 2 & 3 & -6 \\ 3 & -1 & 2 \\ -1 & 3 & 1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{pmatrix} 5000 \\ 2000 \\ 5500 \end{pmatrix}}$$

A X = B

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 2 & 3 & -6 & 5,000 \\ 3 & -1 & 2 & 2,000 \\ -1 & 3 & 1 & 5,500 \end{pmatrix}$	
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 3 & -1 & 2 & 2000 \\ 2 & 3 & -6 & 5000 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 0 & 8 & 5 & 18500 \\ 0 & 9 & -4 & 16000 \end{pmatrix}$	$R_2 \rightarrow R_2 + 3R_1$ $R_3 \rightarrow R_3 + 2R_1$
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 0 & 72 & 45 & 166500 \\ 0 & 72 & -32 & 128000 \end{pmatrix}$	$R_2 \rightarrow 9R_2$ $R_3 \rightarrow 8R_3$
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 0 & 72 & 45 & 166500 \\ 0 & 0 & -77 & -38500 \end{pmatrix}$	$R_3 \rightarrow R_3 - R_2$

$\rho(A) = \rho([A, B]) = 3$ = number of unknowns so the system has unique solution.

$\therefore$ The given system is equivalent to the matrix equation.

$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 72 & 45 \\ 0 & 0 & -77 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5500 \\ 166500 \\ -38500 \end{pmatrix}$$

$$-x + 3y - z = 5500 \quad \text{... (1)}$$
$$72y + 45z = 166500 \quad \text{... (2)}$$
$$-77z = -38500 \quad \text{... (3)}$$

$(3) \Rightarrow \quad -77z = -38500$

$\Rightarrow \quad z = \dfrac{\cancel{-}38500}{\cancel{-}77} = 500$

$(2) \Rightarrow \quad 72y + 45(500) = 166500$

$\Rightarrow \quad 72y + 22,500 = 166500$

$\Rightarrow \quad 72y = 166500 - 22500$

$\Rightarrow \quad 72y = 144000$

$\Rightarrow \quad y = \dfrac{144000}{72} \Rightarrow y = 2,000$

$(1) \Rightarrow -x + 3(2000) + 500 = 5500$

$\Rightarrow \quad -x \cancel{+} 6500 = 5500$

$-x = -5500 - 6500$

$\cancel{-}x = \cancel{-}1000$

$x = 1000$

$\therefore$ The prices per unit of the three commodities are ₹1000, ₹ 2000 and ₹ 500.

8. An amount of ₹ 5,000/- is to be deposited in three different bonds bearing 6%, 7% and 8% per year respectively. Total annual income is ₹358/-. If the income from first two investments is ₹ 70/- more than the income from the third, then find the amount of investment in each bond by rank method.

Sol : Let the amount of investment in each bond be ₹ x, ₹ y and ₹ z respectively.

$$\text{Given } x+y+z = 5000 \quad \dots (1)$$

$$\text{Also } \frac{6x}{100}+\frac{7y}{100}+\frac{8z}{100} = 358$$

$$\boxed{\because \text{ Interest} = \frac{PNR}{100} = \frac{x\times1\times6}{100} = \frac{6x}{100}}$$

$$\Rightarrow \frac{6x+7y+8z}{100} = 358$$

$$\Rightarrow 6x+7y+8z = 35800 \quad \dots (2)$$

$$\text{Given that } \frac{6x}{100}+\frac{7y}{100} = 70+\frac{8z}{100}$$

$$\Rightarrow \frac{6x+7y}{\cancel{100}} = \frac{7000+8z}{\cancel{100}}$$

$$\Rightarrow 6x+7y = 700+8z$$

$$\Rightarrow 6x+7y-8z = 7000 \quad \dots (3)$$

The matrix equation corresponding to the given system is.

$$\begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5000 \\ 35800 \\ 7000 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 5000 \\ 6 & 7 & 8 & 35800 \\ 6 & 7 & -8 & 7000 \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 & 1 & 5000 \\ 0 & 1 & 2 & 5800 \\ 0 & 1 & -14 & -23000 \end{pmatrix}$	$R_2 \to R_2 - 6R_1$ $R_3 \to R_3 - 6R_1$
$\sim\begin{pmatrix} 1 & 1 & 1 & 5000 \\ 0 & 1 & 2 & 5800 \\ 0 & 0 & -16 & -28800 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The last equivalent matrix is in echelon form and $\rho(A) = \rho([A, B]) = 3$ = Number of unknowns.

Thus, the given system is consistent with unique solution. To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -16 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5000 \\ 5800 \\ -28800 \end{pmatrix}$$

$$\Rightarrow x+y+z = 5000 \quad \dots (1)$$

$$\Rightarrow y+2z = 5800 \quad \dots (2)$$

$$\Rightarrow -16z = -28800 \quad \dots (3)$$

$$(3) \Rightarrow -16z = -28800$$

$$\Rightarrow z = -\frac{28800}{-16} = 1800$$

Substituting $z = 1800$ in (2) we get,

$$y+2(1800) = 5800$$

$$\Rightarrow y+3600 = 5800$$

$$\Rightarrow y = 5800-3600$$

$$\Rightarrow y = 2200$$

Substituting $y = 2200$ and $z = 1800$ in (1) we get,

$$x+2200+1800 = 5000$$

$$\Rightarrow x+4000 = 5000$$

$$\Rightarrow x = 5000-4000$$

$$\Rightarrow x = 1000$$

Hence, the amount of investment in each bond is ₹ 1000, ₹ 2200 and ₹1800 respectively.

EXERCISE 1.2

1. Solve the following equations by using Cramer's rule.

(i) $2x+3y=7,\ 3x+5y=9$

(ii) $5x+3y=17$; $3x+7y=31$

(iii) $2x+y-z=3,\ x+y+z=1, x-2y-3z=4$

(iv) $x+y+z=6, 2x+3y-z=5, 6x-2y-3z=-7$

(v) $x+4y+3z=2, 2x-6y+6z=-3,$ $5x-2y+3z=-5$

Sol : (i) $2x+3y=7,\ 3x+5y=9$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10-9 = 1 \neq 0.$$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 7 & 3 \\ 9 & 5 \end{vmatrix} = 7(5)-9\,(3)$$

$$= 35 - 27 = 8$$

$$\Delta y = \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} = 2(9) - 3(7)$$

$$= 18 - 21 = -3$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{8}{1} = 8$$

$$y = \frac{\Delta y}{\Delta} = \frac{-3}{1} = -3$$

$\therefore$ Solution is $x = 8$ and $y = -3$.

(ii) $5x + 3y = 17$; $3x + 7y = 31$ [HY-2019]

Sol : $$\Delta = \begin{vmatrix} 5 & 3 \\ 3 & 7 \end{vmatrix} = 5(7) - 3(3)$$

$$= 35 - 9 = 26$$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 17 & 3 \\ 31 & 7 \end{vmatrix} = 17(7) - 31(3)$$

$$= 119 - 93 = 26$$

$$\Delta y = \begin{vmatrix} 5 & 17 \\ 3 & 31 \end{vmatrix} = 5(31) - 17(3)$$

$$= 155 - 51 = 104.$$

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{26}}{\cancel{26}} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{4}{\cancel{104}}}{\cancel{26}} = 4$$

$\therefore$ Solution is $x = 1$ and $y = 4$.

(iii) $2x + y - z = 3, x + y + z = 1, x - 2y - 3z = 4$

Sol : $$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 2(-3+2) - 1(-3-1) - 1(-2-1)$$

$$= 2(-1) - 1(-4) - 1(-3)$$

$$= -2 + 4 + 3 = 5.$$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$x = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}$$

$$= 3(-3 + 2) - 1(-3 - 4) - 1(-2 - 4)$$

$$= 3(-1) - 1(-7) - 1(-6)$$

$$= -3 + 7 + 6 = 10.$$

$$\Delta y = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= 2(-3 - 4) - 3(-3 - 1) - 1(4 - 1)$$

$$= 2(-7) - 3(-4) - 1(3) = -14 + 12 - 3 = -5$$

$$\Delta z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 2(4 + 2) - 1(4 - 1) + 3(-2 - 1)$$

$$= 2(6) - 1(3) + 3(-3) = 12 - 3 - 9 = 0$$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{\overset{2}{\cancel{10}}}{\cancel{5}} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{\cancel{-5}}{\cancel{5}} = -1$$

$$z = \frac{\Delta z}{\Delta} = \frac{0}{5} = 0$$

$\therefore$ Solution is $(x, y, z) = (2, -1, 0)$

(iv) $x + y + z = 6, 2x + 3y - z = 5, 6x - 2y - 3z = -7.$

Sol : **[PTA - 6; QY -2019]**

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 6 & -2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & -1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 6 & -2 \end{vmatrix}$$

$$= 1(-9 - 2) - 1(-6 + 6) + 1(-4 - 18)$$

$$= 1(-11) - 1(0) + 1(-22)$$

$$= -11 - 22 = -33 \neq 0$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 6 & 1 & 1 \\ 5 & 3 & -1 \\ -7 & -2 & -3 \end{vmatrix} = 6\begin{vmatrix} 3 & -1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 5 & -1 \\ -7 & -3 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ -7 & -2 \end{vmatrix}$$

$$= 6(-9 - 2) - 1(-15 - 7) + 1(-10 + 21)$$

$$= 6(-11) - 1(-22) + 1(11)$$

$$= -66 + 22 + 11 = -33$$

$$\Delta y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 5 & -1 \\ 6 & -7 & -3 \end{vmatrix} = 1\begin{vmatrix} 5 & -1 \\ -7 & -3 \end{vmatrix} - 6\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 1\begin{vmatrix} 2 & 5 \\ 6 & -7 \end{vmatrix}$$

$= 1(-15-7) - 6(-6+6) + 1(-14-30)$
$= 1(-22) - 6(0) + 1(-44)$
$= -22 - 44 = -66$

$$\Delta z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & 5 \\ 6 & -2 & -7 \end{vmatrix} = 1\begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 6 & -7 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 6 & -2 \end{vmatrix}$$

$= 1(-21+10) - 1(-14-30) + 6(-4-18)$
$= 1(-11) - 1(-44) + 6(-22)$
$= -11 + 44 - 132 = -99$

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{-33}}{\cancel{-33}} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{2}{\cancel{-66}}}{\cancel{-33}} = 2$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{3}{\cancel{-99}}}{\cancel{-33}} = 3$$

Hence the solution is $(x, y, z) = (1, 2, 3)$.

(v) $x + 4y + 3z = 2$, $2x - 6y + 6z = -3$, $5x - 2y + 3z = -5$

Sol : $$\Delta = \begin{vmatrix} 1 & 4 & 3 \\ 2 & -6 & 6 \\ 5 & -2 & 3 \end{vmatrix} = 1\begin{vmatrix} -6 & 6 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & 6 \\ 5 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & -6 \\ 5 & -2 \end{vmatrix}$$

$= 1(-18 + 12) - 4(6 - 30) + 3(-4 + 30)$
$= 1(-6) - 4(-24) + 3(26)$
$= -6 + 96 + 78 = 168 \neq 0.$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 2 & 4 & 3 \\ -3 & -6 & 6 \\ -5 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} -6 & 6 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} -3 & 6 \\ -5 & 3 \end{vmatrix} + 3\begin{vmatrix} -3 & -6 \\ -5 & -2 \end{vmatrix}$$

$= 2(-18 + 12) - 4(-9 + 30) + 3(6 - 30)$
$= 2(-6) - 4(21) + 3(-24)$
$= -12 - 84 - 72 = -168$

$$\Delta y = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 6 \\ 5 & -5 & 3 \end{vmatrix} = 1\begin{vmatrix} -3 & 6 \\ -5 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 6 \\ 5 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & -3 \\ 5 & -5 \end{vmatrix}$$

$= 1(-9 + 30) - 2(6 - 30) + 3(-10 + 15)$
$= 1(21) - 2(-24) + 3(5)$
$= 21 + 48 + 15 = 84$

$$\Delta z = \begin{vmatrix} 1 & 4 & 2 \\ 2 & -6 & -3 \\ 5 & -2 & -5 \end{vmatrix} = 1\begin{vmatrix} -6 & -3 \\ -2 & -5 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ 5 & -5 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -2 \end{vmatrix}$$

$= 1(30 - 6) - 4(-10 + 15) + 2(-4 + 30)$
$= 24 - 4(5) + 2(26)$
$= 24 - 20 + 52 = 56$

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{-168}}{\cancel{168}} = -1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\cancel{84}}{\underset{2}{\cancel{168}}} = \frac{1}{2}$$

$$z = \frac{\Delta z}{\Delta} = \frac{\cancel{56}}{\underset{3}{\cancel{168}}} = \frac{1}{3}$$

Hence the solution is $(x, y, z) = \left(-1, \frac{1}{2}, \frac{1}{3}\right)$

2. A commodity was produced by using 3 units of labour and 2 units of capital, the total cost is ₹ 62. If the commodity had been produced by using 4 units of labour and one unit of capital, the cost is ₹ 56. What is the cost per unit of labour and capital? (Use determinant method).

Sol : Let ₹ x represents the cost per unit of labour and ₹ y represents the cost per unit of capital

Given $3x + 2y = 62$
$4x + y = 56$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = 3 - 8 = -5$$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 62 & 2 \\ 56 & 1 \end{vmatrix} = 62(1) - 56(2) = 62 - 112 = -50$$

$$\Delta y = \begin{vmatrix} 3 & 62 \\ 4 & 56 \end{vmatrix} = 3(56) - 4(62)$$

$= 168 - 248 = -80$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{\overset{10}{\cancel{-50}}}{\cancel{-5}} = 10$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{16}{\cancel{-80}}}{\cancel{-5}} = 16.$$

$\therefore$ Cost per unit of labour is ₹ 10 and the cost per unit of capital is ₹16.

3. A total of ₹ 8,600 was invested in two accounts. One account earned $4\frac{3}{4}$ % annual interest and the other earned $6\frac{1}{2}$ % annual interest. If the total interest for one year was ₹ 431.25, how much was invested in each account? (Use determinant method).

Sol : Let the amount invested in the two accounts be ₹ x and ₹ y respectively

By the given data, $x + y = 8600$ ... (1)

$$4\frac{3}{4}\times\frac{x}{100}+6\frac{1}{2}\times\frac{y}{100} = 431.25$$

$$\left[\therefore \text{ Interest} = \frac{PNR}{100}\right]$$

$$\Rightarrow \frac{19x}{400}+\frac{13y}{200} = 431.25$$

$$\Rightarrow \frac{19x+26y}{400} = 431.25$$

$$19x + 26y = 172500 \quad ... (2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 19 & 26 \end{vmatrix} = 1(26) - 1(19) = 26 - 19 = 7$$

$$\Delta x = \begin{vmatrix} 8600 & 1 \\ 172500 & 26 \end{vmatrix} = 8600(26) - 1(172500) = 223600 - 172500 = 51100$$

$$\Delta y = \begin{vmatrix} 1 & 8600 \\ 19 & 172500 \end{vmatrix} = 1(172500) - 19(8600) = 172500 - 163400 = 9100$$

By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{51100}{7} = 7300$$

$$y = \frac{\Delta y}{\Delta} = \frac{9100}{7} = 1300$$

Hence the amount invested at $4\frac{3}{4}$% is ₹7300 and amount invested at $6\frac{1}{2}$% is ₹1300.

4. At marina two types of games viz., Horse riding and Quad Bikes riding are available on hourly rent. Keren and Benita spent ₹ 780 and ₹ 560 during the month of May. [PTA-3]

Name	Number of Hours		Total amount spent (in ₹)
	Horse Riding	Quad Bike Riding	
Keren	3	4	780
Benita	2	3	560

Find the hourly charges for the two games (rides). (Use determinant method).

Sol : Let the hourly charges for horse riding be ₹ x and the hourly charges for quad bike be ₹ y from the given data, $3x + 4y = 780$

$$2x + 3y = 560$$

$$\Delta = \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 3(3) - 2(4) = 9 - 8 = 1 \neq 0$$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 780 & 4 \\ 560 & 3 \end{vmatrix} = 780(3) - 4(560) = 2340 - 2240 = 100$$

$$\Delta y = \begin{vmatrix} 3 & 780 \\ 2 & 560 \end{vmatrix} = 3(560) - 2(780) = 1680 - 1560 = 120$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{100}{1} = 100$$

$$y = \frac{\Delta y}{\Delta} = \frac{120}{1} = 120$$

$\therefore$ Hourly charges for the two rides are ₹100 and ₹120 respectively.

5. In a market survey three commodities A, B and C were considered. In finding out the index number some fixed weights were assigned to the three varieties in each of the commodities. The table below provides the information regarding the consumption of three commodities according to the three varieties and also the total weight received by the commodity

Commodity Variety	Variety			Total Weight
	I	II	III	
A	1	2	3	11
B	2	4	5	21
C	3	5	6	27

Find the weights assigned to the three varieties by using Cramer's Rule.

Sol : Let the weight assigned to the three varieties be ₹ x, ₹y and ₹z respectively.

By the given data,

$$x + 2y + 3z = 11$$
$$2x + 4y + 5z = 21$$
$$3x + 5y + 6z = 27$$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = 1\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$= 1(24 - 25) - 2(12 - 15) + 3(10 - 12)$

$= 1(-1) - 2(-3) + 3(-2)$

$= -1 + 6 - 6 = -1 \neq 0.$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 11 & 2 & 3 \\ 21 & 4 & 5 \\ 27 & 5 & 6 \end{vmatrix} = 11\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 21 & 5 \\ 27 & 6 \end{vmatrix} + 3\begin{vmatrix} 21 & 4 \\ 27 & 5 \end{vmatrix}$$

$= 11(24 - 25) - 2(126 - 135) + 3(105 - 108)$

$= 11(-1) - 2(-9) + 3(-3) = -11 + 18 - 9 = -2$

$$\Delta y = \begin{vmatrix} 1 & 11 & 3 \\ 2 & 21 & 5 \\ 3 & 27 & 6 \end{vmatrix} = \begin{vmatrix} 21 & 5 \\ 27 & 6 \end{vmatrix} - 11\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 21 \\ 3 & 27 \end{vmatrix}$$

$= 1(126 - 135) - 11(12 - 15) + 3(54 - 63)$

$= -9 - 11(-3) + 3(-9) = -9 + 33 - 27 = -3$

$$\Delta z = \begin{vmatrix} 1 & 2 & 11 \\ 2 & 4 & 21 \\ 3 & 5 & 27 \end{vmatrix} = 1\begin{vmatrix} 4 & 21 \\ 5 & 27 \end{vmatrix} - 2\begin{vmatrix} 2 & 21 \\ 3 & 27 \end{vmatrix} + 11\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$= 1(108 - 105) - 2(54 - 63) + 11(10 - 12)$

$= 1(3) - 2(-9) + 11(-2) = 3 + 18 - 22 = -1$

$$x = \frac{\Delta x}{\Delta} = \frac{-2}{-1} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{-3}{-1} = 3$$

$$\text{and } z = \frac{\Delta z}{\Delta} = \frac{-1}{-1} = 1$$

Hence, the weights assigned to the three varieties are 2, 3 and 1 respectively.

6. A total of ₹ 8,500 was invested in three interest earning accounts. The interest rates were 2%, 3% and 6% if the total simple interest for one year was ₹ 380 and the amount invested at 6% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account? (Use Cramer's rule).

[PTA-2]

Sol : Let the amount invested in the rate of 2%, 3% and 6% be ₹ x, ₹ y and ₹ z respectively.

By the given data,

$$x + y + z = 8500 \qquad \ldots (1)$$

$$\frac{2x}{100} + \frac{3y}{100} + \frac{6z}{100} = 380$$

$$\Rightarrow \frac{2x + 3y + 6z}{100} = 380$$

$$\boxed{\because \text{Interest} = \frac{\text{PNR}}{100} = \frac{x \times 1 \times 2}{100} = \frac{2x}{100}}$$

$$\Rightarrow 2x + 3y + 6z = 38000 \qquad \ldots (2)$$

Also, $z = x + y$

$$x + y - z = 0 \qquad \ldots (3)$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 6 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$$

$= 1(-3 - 6) - 1(-2 - 6) + 1(2 - 3)$

$= 1(-9) - 1(-8) + 1(-1) = -9 + 8 - 1 = -2 \neq 0$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 8500 & 1 & 1 \\ 38000 & 3 & 6 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 8500\begin{vmatrix} 3 & 6 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 38000 & 6 \\ 0 & -1 \end{vmatrix} + 1\begin{vmatrix} 38000 & 3 \\ 0 & 1 \end{vmatrix}$$

$= 8500(-3 - 6) - 1(-38000 - 0) + 1(38000 - 0)$

$= 8500(-9) - 1(-38000) + 1(38000)$

$= -76500 + 38000 + 38000 = -500$

$$\Delta y = \begin{vmatrix} 1 & 8500 & 1 \\ 2 & 38000 & 6 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 38000 & 6 \\ 0 & -1 \end{vmatrix} - 8500\begin{vmatrix} 2 & 6 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 38000 \\ 1 & 0 \end{vmatrix}$$

$= 1(-38000 - 0) - 8500(-2 - 6) + 1(0 - 38000)$

$= -38000 - 8500(-8) - 38000$

$= -38000 + 68000 - 38000 = -8000$

$$\Delta z = \begin{vmatrix} 1 & 1 & 8500 \\ 2 & 3 & 38000 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 38000 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 38000 \\ 1 & 0 \end{vmatrix} + 8500\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$$

$= 1(0 - 38000) - 1(0 - 38000) + 85000(2 - 3)$

$= -38000 + 38000 + 8500(-1) = -8500$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{\overset{250}{\cancel{-500}}}{\cancel{-2}} = +250$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{4000}{\cancel{-8000}}}{\cancel{-2}} = 4000$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{4250}{\cancel{-8500}}}{\cancel{-2}} = 4250$$

Thus, the amount invested at 2% is ₹250 at 3% is ₹4000 and at 6% ₹ 4250.

EXERCISE 1.3

1. The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 45% of those who already subscribe will subscribe again while 30% of those who do not now subscribe will subscribe. On the last letter, it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

Sol : Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.45 & 0.55 \\ 0.30 & 0.70 \end{pmatrix} \end{array}$$

Where A represents the percentage of subscribers and B represents the percentage of non-subscribers. By the given data, 40% received the order of subscription $\Rightarrow$ 60% are non-subscribers.

$$A = 40\% = 0{\cdot}40$$
$$\text{and } B = 60\% = 0{\cdot}60$$

$$\therefore \begin{array}{cc} A & B \end{array} \atop (0.40 \quad 0.60) \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.45 & 0.55 \\ 0.30 & 0.70 \end{pmatrix} \end{array} =$$

$$\big((0.40)(0.45)+(0.60)(0.30) \quad (0.40)(0.55)+(0.60)(0.70)\big)$$

$$= (0.18+0.18 \quad 0.22+0.42)$$
$$= (0{\cdot}36 \quad 0{\cdot}64)$$

$\Rightarrow$ 36 % of those receiving the current letter can be expected to order a subscription.

2. A new transit system has just gone into operation in Chennai. Of those who use the transit system this year, 30% will switch over to using metro train next year and 70% will continue to use the transit system. Of those who use metro train this year, 70% will continue to use metro train next year and 30% will switch over to the transit system. Suppose the population of Chennai city remains constant and that 60% of the commuters use the transit system and 40% of the commuters use metro train next year.

(i) What percent of commuters will be using the transit system after the next year?

(ii) What percent of commuters will be using the transit system in the long run? [HY-2019]

Sol : Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.70 & 0.30 \\ 0.30 & 0.70 \end{pmatrix} \end{array}$$

Where A represents the percentage of people using transit system and B represents the percentage of people using metro train.

By the given data

$$A = 60\% = 0{\cdot}60$$
$$\text{and } B = 40\% = 0{\cdot}40$$

$$(0.60 \quad 0.40)\begin{pmatrix} 0.70 & 0.30 \\ 0.30 & 0.70 \end{pmatrix}$$
$$= \big((0.6)(0.7)+(0.4)(0.3) \quad (0.6)(0.3)+(0.4)(0.7)\big)$$
$$= (0.42+0.12 \quad 0.18+0.28) = (0.54 \quad 0.46)$$

$\therefore$ A = 54% and B = 46%

(i) The percent of Commuters using the transit system after one year is 54% and the percent of commuters using the metro train after the next year is 46%

(ii) Equilibrium will be reached in the long run. At equilibrium we must have

$$(A \quad B)\, T = (A \quad B)$$

where $A + B = 1$

$$\Rightarrow (A \quad B)\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} = (A \quad B)$$

$$\Rightarrow (0.7A + 0.3B \quad 0.3A + 0.7B) = (A \quad B)$$

Equaling the entries on both sides, we get

$$0.7A + 0.3B = A$$
$$\Rightarrow {\cdot}7A + 0.3\,(1 - A) = A$$
$$[\because A + B = 1 \Rightarrow B = 1 - A]$$
$$\Rightarrow 0.7A + 0.3 - 0.3A = A$$
$$\Rightarrow 0.3 = A - 0.7A + 0.3A$$
$$\Rightarrow 0.3 = A\,(1 - 0.7 + 0.3)$$
$$\Rightarrow 0.3 = A\,(0.3 + 0.3)$$
$$\Rightarrow 0.3 = A\,(0.6)$$
$$\Rightarrow A = \frac{0.3}{0.6} = \frac{1}{2} = 0{\cdot}50$$

$\therefore$ The percent of commuters using the transit system in the long run is 50%.

3. Two types of soaps A and B are in the market. Their present market shares are 15% for A and 85% for B. Of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after one year and when is the equilibrium reached?

[PTA-2; Govt. MQP & QY- 2019]

Sol : Transition probability matrix

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{pmatrix} \end{array}$$

Where A represents the percent of people those who bought soap A and B represents the percent of people those who bought soap B.

By the given data,

$$A = 15\% = 0{\cdot}15$$
$$\text{and } B = 85\% = 0{\cdot}85$$

Percentage after one year is

$$(0.15 \quad 0.85)\begin{pmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{pmatrix}$$
$$= ((0.15)(0.65) + (0.85)(0.45) \quad 0.15(0.35) + 0.85(0.55))$$
$$= (0.0975 + 0.3825 \quad 0.0525 + 0.4675) = (0.48 \quad 0.52)$$

Hence, market share after one year is 48% and 52%

At equilibrium,

$$(A \quad B)\, T = (A \quad B)$$
$$(A \quad B)\begin{pmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{pmatrix} = (A \quad B)$$
$$(0.65A + 0.45B \quad 0.35A + 0.55B) = (A \quad B)$$

Equating the corresponding entries on both sides we get,

$$0.65A + 0.45B = A$$
$$\Rightarrow 0.65A + 0.45(1 - A) = A$$
$$[\text{Since } A + B = 1 \Rightarrow B = 1 - A]$$
$$\Rightarrow 0.65A + 0.45 - 0.45A = A$$
$$\Rightarrow 0.45 = A - 0.65A + 0.45A$$
$$\Rightarrow 0.45 = A(1 - 0.65 + 0.45)$$
$$\Rightarrow 0.45 = A(0.35 + 0.45)$$
$$\Rightarrow 0.45 = A(0.8)$$
$$\Rightarrow A = \frac{0.45}{0.8} = 0.5625 = 56.25\%$$
$$\therefore B = 1 - A = 1 - 0.5625 = 0.4375 = 43.75\%$$

$\therefore$ Equilibrium is reached when A = 56.25% and B = 43.75%

4. Two products A and B currently share the market with shares 50% and 50% each respectively. Each week some brand switching takes place. Of those who bought A the previous week, 60% buy it again whereas 40% switch over to B. Of those who bought B the previous week, 80% buy it again where as 20% switch over to A. Find their shares after one week and after two weeks. If the price war continues, when is the equilibrium reached?

[Sep. - 2020]

Sol : Transition probability matrix

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

By the given data

$$A = 50\% = 0.5$$
$$B = 50\% = 0.5$$

Shares after one week

$$(0.5 \quad 0.5)\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$
$$= ((0.5)(0.6) + (0.5)(0.2) \quad 0.5(0.4) + 0.5(0.8))$$
$$= (0.30 + 0.10 \quad 0.20 + 0.40) = (0.40 \quad 0.60)$$

$\therefore$ Shares after one week for products A and B are 40% and 60% respectively.

Shares after two weeks

$$(0.4 \quad 0.6)\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$
$$= ((0.4)(0.6) + (0.6)(0.2) \quad (0.4)(0.4) + 0.6(0.8))$$
$$= (0.24 + 0.12 \quad 0.16 + 0.48) = 0.36 \quad 0.64)$$

$\therefore$ Shares after two week for products A and B are 36% and 64% respectively.

At equilibrium, we must have

$$(A \quad B)\, T = (A \quad B)$$
$$\text{where } A + B = 1$$
$$(A \quad B)\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = (A \quad B)$$
$$\Rightarrow (0.6A + 0.2B \quad 0.4A + 0.8B) = (A \quad B)$$

Equating the corresponding entries on both sides we get,

$$0.6A + 0.2B = A$$
$$\Rightarrow 0.6A + 0.2(1 - A) = A$$
$$\Rightarrow 0.6A + 0.2 - 0.2A = A$$
$$\Rightarrow 0.2 = A - 0.6A + 0.2A$$
$$\Rightarrow 0.2 = A(1 - 0.6 + 0.2)$$
$$\Rightarrow 0.2 = A(0.4 + 0.2)$$
$$\Rightarrow 0.2 = A(0.6)$$
$$\Rightarrow A = \frac{0.2}{0.6} = 0.33$$
$$\Rightarrow A = 33\%$$

and $B = 1 - A = 1 - 0.33 = 0.67 \Rightarrow B = 67\%$

$\therefore$ Equilibrium is reached when A = 33% and B = 67%

EXERCISE 1.4

Choose the correct answer

1. If A = (1 2 3), then the rank of AA^T is

(a) 0 (b) 2 (c) 3 (d) 1

[Ans: (d) 1]

Hint: $AA^T = \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = (1+4+9) = 14$

$\therefore$ Rank of AA^T is 1

2. The rank of $m \times n$ matrix whose elements are unity is

(a) 0 (b) 1 (c) m (d) n

[Ans: (b) 1]

Hint: $A = \begin{pmatrix}1 & 1 & 1 & 1 & 1 & --- & 1\\ 1 & 1 & 1 & 1 & 1 & --- & 1\\ - & - & - & - & - & --- & -\\ - & - & - & - & - & --- & -\\ 1 & 1 & 1 & 1 & 1 & 1 & 1\end{pmatrix}$

m rows and n columns applying the elementary transformation.

$R_2 \to R_2 - R_1, R_3 \to R_1 \ldots R_m \to R_m - R_1$ we get

$A = \begin{pmatrix}1 & 1 & 1 & --- & --- & 1\\ 0 & 0 & 0 & --- & --- & 0\\ 0 & 0 & 0 & --- & --- & 0\\ - & - & - & --- & --- & -\\ - & - & - & - & - & -\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$

$\therefore$ Rank is 1

3. If T = $\begin{array}{c|c} & \begin{matrix}A & B\end{matrix}\\ \begin{matrix}A\\B\end{matrix} & \begin{pmatrix}0.4 & 0.6\\ 0.2 & 0.8\end{pmatrix}\end{array}$ is a transition probability matrix, then at equilibrium A is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{1}{6}$ (d) $\frac{1}{8}$

[Ans: (a) $\frac{1}{4}$]

Hint: At equilibrium,

$$\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}0\cdot4 & 0\cdot6\\ 0\cdot2 & 0\cdot8\end{pmatrix} = \begin{pmatrix}A & B\end{pmatrix}$$

Where $A + B = 1$

$$\begin{pmatrix}0\cdot4A+0\cdot2B & 0\cdot6A+0\cdot8B\end{pmatrix} = \begin{pmatrix}A & B\end{pmatrix}$$

$0\cdot4A + 0\cdot2B = A$

$0\cdot4A + 0\cdot2(1-A) = A$

[since $A+B = 1 \Rightarrow B = 1-A$]

$0\cdot4A + 0\cdot2 - 0.2A = A$

$\Rightarrow 0\cdot2 = A - 0\cdot4A + 0\cdot2A$

$\Rightarrow 0\cdot2 = A(1-0\cdot4+0\cdot2)$

$= A(0\cdot8)$

$\Rightarrow A = \frac{0\cdot2}{0\cdot8} = \frac{1}{4}$

4. If A = $\begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix}$ then ρ (A) is

(a) 0 (b) 1 (c) 2 (d) n

[Ans: (c) 2]

Hint: If A = $\begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix}$

$|A| = \begin{vmatrix}2 & 0\\ 0 & 8\end{vmatrix} = 16 - 0 = 16 \neq 0.$

Since the second order determinant does not vanish. ρ (A) = 2

5. The rank of the matrix $\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix}$ is

(a) 0 (b) 1 (c) 2 (d)3

[Ans: (d) 3]

Hint: Let A= $\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 3 & 8\end{pmatrix} \begin{matrix}R_2 \to R_2 - R\\ R_3 \to R_3 - R\end{matrix}$

$= \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 2\end{pmatrix} R_3 \to R_3 - 3R_2$

Since there are 3 non-zero rows, ρ (A) = 3

$\therefore \rho$ (A) = 3

6. The rank of the unit matrix of order n is

(a) $n-1$ (b) n (c) $n+1$ (d) n^2

[PTA - 3; QY - 2019]

[Ans: (b) n]

Hint: The rank of the unit matrix of order n

[Since the rank of a non-singular matrix of order $n \times n$ is 'n']

7. If ρ(A) = r then which of the following is correct? **[Sep-2020]**

(a) all the minors of order r which does not vanish.

(b) A has at least one minor of order r which does not vanish.

(c) A has at least one $(r + 1)$ order minor which vanishes.

(d) all $(r + 1)$ and higher order minors should not vanish.

[Ans: (b) A has at least one minor of order r which does not vanish.]

8. If $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then the rank of AA^T is

(a) 0 (b) 1 (c) 2 (d) 3

[Ans: (b) 1]

Hint: If $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $A^T = (1\ 2\ 3)$

$$AA^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (1\ 2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \therefore \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Rank of AA^T is 1

9. If the rank of the matrix $\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{pmatrix}$ is 2, then λ is **[Govt. MQP - 2019]**

(a) 1 (b) 2

(c) 3 (d) only real number

[Ans: (a) 1]

Hint: Rank of $\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{pmatrix} = 2$

Since the rank is 2,

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & \lambda \\ -1 & 0 \end{vmatrix} = 0$$

$$\lambda(\lambda^2) + 1(0 - 1) + 0 = 0$$

$$\lambda^3 - 1 = 0 \Rightarrow \lambda^3 = 1$$

$$\lambda = 1$$

10. The rank of the diagonal matrix $\begin{pmatrix} 1 & & & & & \\ & 2 & & & & \\ & & -3 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$

(a) 0 (b) 2 (c) 3 (d) 5

[Ans: (c) 3]

Hint: Since there are 3 non-zero rows, rank is 3

11. If $T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0\cdot7 & 0\cdot3 \\ 0\cdot6 & x \end{pmatrix} \end{array}$ is a transition probability matrix, then the value of x is **[PTA-6; QY-2019]**

(a) 0.2 (b) 0.3 (c) 0.4 (d) 0.7

[Ans: (c) 0·4]

Hint: $T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0\cdot7 & 0\cdot3 \\ 0\cdot6 & x \end{pmatrix} \end{array}$

Since this is a transition probability matrix,

$$0\cdot6 + x = 1 \Rightarrow x = 1 - 0\cdot6 = 0\cdot4$$

12. Which of the following is not an elementary transformation? **[March - 2020]**

(a) $R_i \leftrightarrow R_j$ (b) $R_i \to 2R_i + 2C_j$

(c) $R_i \to 2R_i - 4R_j$ (d) $C_i \to C_i + 5C_j$

[Ans: (b) $R_i \to 2R_i + 2C_j$]

Hint: $R_i \to 2R_i + 2C_j$ is not an elementary transformation since it includes rows and columns.

13. If $\rho(A) = \rho(A, B)$ then the system is **[PTA-4]**

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) Consistent (d) inconsistent

[Ans: (c) Consistent]

Hint: If $\rho(A) = \rho(A, B)$ then the system can have a unique solution or infinitely many solutions.
$\therefore$ The system is consistent.

14. If $\rho(A) = \rho(A, B)$ = the number of unknowns, then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) inconsistent (d) consistent

[Ans: (b) Consistent and has a unique solution]

Hint: If $\rho(A) = \rho(A, B)$ = Number of unknowns then the system is consistent and has a unique solution.

15. If $\rho(A) \neq \rho(A, B)$, then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) inconsistent (d) consistent

[Ans: (c) inconsistent]

Hint: If $\rho(A) \neq \rho(A, B)$, then the system is inconsistent.

16. In a transition probability matrix, all the entries are greater than or equal to

(a) 2 (b) 1 (c) 0 (d) 3

[Ans: (c) 0]

Hint: $[\because 0 < p < 1]$

17. If the number of variables in a non-homogeneous system AX = B is *n*, then the system possesses a unique solution only when **[HY-2019]**

(a) $\rho(A) = \rho(A, B) > n$

(b) $\rho(A) = \rho(A, B) = n$

(c) $\rho(A) = \rho(A, B) < n$ (d) none of these

[Ans: (b) $\rho(A) = \rho(A, B) = n$]

18. The system of equations $4x + 6y = 5$, $6x + 9y = 7$ has **[PTA-3; Govt. MQP - 2019]**

(a) a unique solution (b) no solution

(c) infinitely many solutions (d) none of these

[Ans: (b) no solution]

Hint: Given
$$4x + 6y = 5$$
$$6x + 9y = 7$$
$$[A, B] = \begin{pmatrix} 4 & 6 & 5 \\ 6 & 9 & 7 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & \frac{6}{4} & \frac{5}{4} \\ 6 & 9 & 7 \end{pmatrix} R_1 \to R_1 \div 4$$
$$\sim \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{4} \\ 0 & 0 & \frac{-1}{2} \end{pmatrix} R_2 \to R_2 - 6R_1$$

Here $\rho(A) = 1$ and $\rho(A, B) = 2$

Since $\rho(A) \neq \rho(A, B)$, the system has no solution

19. For the system of equations $x + 2y + 3z = 1$, $2x + y + 3z = 2$ $5x + 5y + 9z = 4$

(a) there is only one solution

(b) there exists infinitely many solutions

(c) there is no solution (d) None of these

[Ans: (a) there is only one solution.]

Hint: Given $x + 2y + 3z = 1$, $2x + y + 3z = 2$, $5x + 5y + 9z = 4$
$$[A, B] = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 2 \\ 5 & 5 & 9 & 4 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -5 & -6 & -1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$
$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -5 & -6 & -1 \end{pmatrix} R_2 \to R_2 \div 5$$
$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} R_3 \to R_3 - 5R_2$$

Here $\rho(A) = 3$ and $\rho(A, B) = 3$

$\therefore \rho(A) = \rho(A, B) = 3 =$ Number of unknowns.

20. If $|A| \neq 0$, then A is

(a) non- singular matrix (b) singular matrix

(c) zero matrix (d) none of these

[Ans: (a) non- singular matrix]

21. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + k = 4$ has unique solution, if *k* is not equal to **[March -2020]**

(a) 4 (b) 0 (c) –4 (d) 1

[Ans: (b) 0]

Hint: Given $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + k = 4$
$$[A, B] = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & k & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & -1 & k-3 & -2 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & k & -1 \end{pmatrix} R_3 \to R_3 - R_2$$

For unique solution $\rho(A) = \rho(A, B) = 3$

This can happen only when $k \neq 0$

22. Cramer's rule is applicable only to get an unique solution when

(a) $\Delta_z \neq 0$ (b) $\Delta_x \neq 0$

(c) $\Delta \neq 0$ (d) $\Delta_y \neq 0$

[Ans: (c) $\Delta \neq 0$]

23. If $\frac{a_1}{x} + \frac{b_1}{y} = c_1$, $\frac{a_2}{x} + \frac{b_2}{y} = c_2$, $\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$; $\Delta_3 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ then (x, y) is

(a) $\left(\frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_1}\right)$ (b) $\left(\frac{\Delta_3}{\Delta_1}, \frac{\Delta_2}{\Delta_1}\right)$

(c) $\left(\frac{\Delta_1}{\Delta_2}, \frac{\Delta_1}{\Delta_3}\right)$ (d) $\left(\frac{-\Delta_1}{\Delta_2}, \frac{-\Delta_1}{\Delta_3}\right)$

[Ans: (d) $\left(\frac{-\Delta_1}{\Delta_2}, \frac{-\Delta_1}{\Delta_3}\right)$]

Hint: Given $\frac{a_1}{x}+\frac{b_1}{y}=c_1;\ \frac{a_2}{x}+\frac{b_2}{y}=c_2$

and $\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix};\ \Delta_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix};\ \Delta_3 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$

$\Rightarrow \quad \Delta_2 = -\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ $[\therefore C_1 \leftrightarrow C_2]$

$\Rightarrow \quad -\Delta_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

and $\Delta_3 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ $C_1 \leftrightarrow C_2$

$\therefore \quad \frac{1}{x} = \frac{-\Delta_2}{\Delta_1}$ and $\frac{1}{y} = \frac{-\Delta_3}{+\Delta_1}$

$\Rightarrow \quad x = \frac{-\Delta_1}{\Delta_2}$ and $y = \frac{-\Delta_1}{\Delta_3}$

24. If $|A_{n\times n}| = 3$ and $|\text{adj}A|=243$ then the value n is [PTA-5; QY-2019]

(a) 4 (b) 5 (c) 6 (d) 7

[Ans: (c) 6]

Hint: Given $|A_{n\times n}| = 3$, $|\text{adj } A| = 243$.

If A is a square matrix of order n, then

$$|\text{adj } A| = |A|^{n-1}$$
$$243 = 3^{n-1}$$
$$3^5 = 3^{n-1}$$
$$5 = n-1$$
$$n = 5+1 = 6$$

25. Rank of a null matrix is [PTA-2]

(a) 0 (b) –1 (c) ∞ (d) 1

[Ans: (a) 0]

Miscellaneous problems

1. Find the rank of the matrix $A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}$ [PTA-2]

Sol : Given $A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -3 & 4 & 0 \\ 0 & 28 & -34 & -63 \end{pmatrix}$ $R_2 \to R_2 - 9R_1$

$\sim \begin{pmatrix} 1 & -3 & 4 & 0 \\ 0 & 0 & \frac{10}{3} & -63 \end{pmatrix}$ $R_2 \to R_2 + \frac{28}{3}.R_1$

The last equivalent matrix is in echelon form and there are 2 non-zero rows.

$\therefore \quad \rho(A) = 2.$

2. Find the rank of the matrix $A = \begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix}$

Sol : Given $A = \begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ -2 & 1 & 3 & 4 \end{pmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 7 & 11 & 18 \end{pmatrix}$ $R_2 \to R_2 + 2R_1$

$\sim \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 4 \end{pmatrix}$ $R_3 \to R_3 - 7R_2$

The last equivalent matrix is in echelon form and there are 3 non - zero rows.

$\therefore \quad \rho(A) = 3.$

3. Find the rank of the matrix $A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$

Sol : Given $A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ $R_3 \to R_3 \div 4$

$\sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & 6 \\ 4 & 5 & 2 & 2 \end{pmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 1 & -6 & 2 \end{pmatrix}$ $\begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$

$\sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 0 & -11 & 8 \end{pmatrix}$ $R_3 \to R_3 + R_2$

The last equivalent matrix is in echelon form and there are 3 non-zero rows.

$\therefore \rho(A) = 3.$

4. Examine the consistency of the system of equations:
$x+y+z=7,\ x+2y+3z=18,\ y+2z=6$.

Sol : Given non homogeneous equations are
$x+y+z=7,\ x+2y+3z=18,\ y+2z=6.$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 18 \\ 0 & 1 & 2 & 6 \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 11 \\ 0 & 1 & 2 & 6 \end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim\begin{pmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & -5 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Here $\rho(A) = 2$ and $\rho(A, B) = 3$

Since $\rho(A) \neq \rho(A, B)$, the given system is inconsistent and has no solution.

5. Find k if the equations $2x + 3y - z = 5$, $3x - y + 4z = 2$, $x + 7y - 6z = k$ are consistent.

Sol : Given non-homogeneous equations are

$2x + 3y - z = 5,\ 3x - y + 4z = 2,\ x + 7y - 6z = k$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 2 & 3 & -1 & 5 \\ 3 & -1 & 4 & 2 \\ 1 & 7 & -6 & k \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 7 & -6 & k \\ 3 & -1 & 4 & 2 \\ 2 & 3 & -1 & 5 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim\begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & -11 & 11 & 5-2k \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim\begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & 0 & 0 & 2(5-2k)-(2-3k) \end{pmatrix}$	$R_3 \to 2R_3 - R_2$
$\sim\begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & 0 & 0 & 10-4k-2+3k \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & 0 & 0 & 8-k \end{pmatrix}$	

Here $\rho(A) = 2$

Since the given system is consistent, $\rho(A, B)$ must be equal to 2.

This can happen only when

$8 - k = 0 \qquad \Rightarrow k = 8$

6. Find k if the equations $x + y + z = 1$, $3x-y-z = 4$, $x + 5y + 5z = k$ are inconsistent.

Sol : $x + y + z = 1,\ 3x - y - z = 4,\ x + 5y + 5z = k$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & k \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & k-1 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - R_1$
$\sim\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & k \end{pmatrix}$	$R_3 \to R_3 + R_2$

Here clearly $\rho(A) = 2$.

Since the given system is inconsistent,

$\rho(A) \neq \rho(A, B)$

This can happen only when $k \neq 0$.

$\therefore$ k can take any value other than zero.

7. Solve the equations $x + 2y + z = 7$, $2x - y + 2z = 4$, $x + y - 2z = -1$ by using Cramer's rule.

Sol : $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

$= 1(2-2) - 2(-4-2) + 1(2+1)$

$= 1(0) - 2(-6) + 1(3) = 12 + 3 = 15 \neq 0.$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$\Delta x = \begin{vmatrix} 7 & 2 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -2 \end{vmatrix} = 7\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 4 & 2 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix}$

$= 7(2-2) - 2(-8+2) + 1(4-1)$

$= 7(0) - 2(-6) + 1(3) = 12 + 3 = 15$

$\Delta y = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & -2 \end{vmatrix} = 1\begin{vmatrix} 4 & 2 \\ -1 & -2 \end{vmatrix} - 7\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix}$

$= 1(-8+2) - 7(-4-2) + 1(-2-4)$

$= 1(-6) - 7(-6) + 1(-6) = -6 + 42 - 6 = 30$

$\Delta z = \begin{vmatrix} 1 & 2 & 7 \\ 2 & -1 & 4 \\ 1 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} + 7\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

$= 1\,(1-4) - 2(-2-4) + 7(2+1)$

$= 1\,(-3) - 2\,(-6) + 7\,(3) = -3 + 12 + 21 = 30$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{15}}{\cancel{15}} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{2}{\cancel{30}}}{\cancel{15}} = 2$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{2}{\cancel{30}}}{\cancel{15}} = 2$$

$\therefore$ The solution is $(x, y, z) = (1, 2, 2)$

8. The cost of 2kg. of wheat and 1kg. of sugar is ₹100. The cost of 1kg. of wheat and 1kg. of rice is ₹80. The cost of 3kg. of wheat, 2kg. of sugar and 1kg of rice is ₹220. Find the cost of each per kg., using Cramer's rule.

Sol : Let the cost of 1kg of wheat be ₹ x, 1kg of sugar be ₹ y and 1kg of rice be ₹ z.

By the given data,

$$2x + y = 100$$
$$x + z = 80$$
$$3x + 2y + z = 220$$

$$\Delta = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$

$$= 2\,(0-2) - 1\,(1-3) + 0\,(2-0)$$

$$= 2(-2) - 1(-2)$$

$$= -4 + 2 = -2 = -2 \neq 0$$

$$\Delta x = \begin{vmatrix} 100 & 1 & 0 \\ 80 & 0 & 1 \\ 220 & 2 & 1 \end{vmatrix} = 100\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 80 & 1 \\ 220 & 1 \end{vmatrix} + 0$$

$$= 100(0-2) - 1\,(80-220)$$

$$= 100(-2) - 1(-140)$$

$$= -200 + 140 = -60.$$

$$\Delta y = \begin{vmatrix} 2 & 100 & 0 \\ 1 & 80 & 1 \\ 3 & 220 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 80 & 1 \\ 220 & 1 \end{vmatrix} - 100\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0$$

$$= 2\,(80-220) - 100\,(1-3)$$

$$= 2\,(-140) - 100\,(-2)$$

$$= -280 + 200 = -80.$$

$$\Delta z = \begin{vmatrix} 2 & 1 & 100 \\ 1 & 0 & 80 \\ 3 & 2 & 220 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 80 \\ 2 & 220 \end{vmatrix} - 1\begin{vmatrix} 1 & 80 \\ 3 & 220 \end{vmatrix} + 100\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$

$$= 2(0-160) - 1(220-240) + 100(2-0)$$

$$= 2(-160) - 1(-20) + 100(2)$$

$$= -320 + 20 + 200 = -100$$

$$x = \frac{\Delta x}{\Delta} = \frac{-60}{-2} = 30$$

$$y = \frac{\Delta y}{\Delta} = \frac{-80}{-2} = 40$$

$$z = \frac{\Delta z}{\Delta} = \frac{-100}{-2} = 50$$

$\therefore$ The cost of 1kg of wheat is ₹. 30

The cost of 1kg sugar is ₹. 40 and

The cost of 1 kg of rice is ₹. 50.

9. A salesman has the following record of sales during three months for three items A,B and C, which have different rates of commission.

Months	Sales of units			Total commission drawn (in ₹)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rate of commission on the items A,B and C by using Cramer's rule.

Sol : Let the rate of commission on the items A, B and C be x, y and z respectively.

By the given data, the non-homogeneous equations are

$$90x + 100y + 20z = 800$$
$$\Rightarrow \quad 9x + 10y + 2z = 80$$
$$130x + 50y + 40z = 900$$
$$\Rightarrow \quad 13x + 5y + 4z = 90$$
$$60x + 100y + 30z = 850$$
$$\Rightarrow \quad 6x + 10y + 3z = 85$$

$$\Delta = \begin{vmatrix} 9 & 10 & 2 \\ 13 & 5 & 4 \\ 6 & 10 & 3 \end{vmatrix} = 9\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} - 10\begin{vmatrix} 13 & 4 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 13 & 5 \\ 6 & 10 \end{vmatrix}$$

$= 9\,(15 - 40) - 10\,(39 - 24) + 2(130 - 30)$

$= 9\,(-25) - 10(15) + 2(100)$

$= -225 - 150 + 200 = -175$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 80 & 10 & 2 \\ 90 & 5 & 4 \\ 85 & 10 & 3 \end{vmatrix} = 80\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} - 10\begin{vmatrix} 90 & 4 \\ 85 & 3 \end{vmatrix} + 2\begin{vmatrix} 90 & 5 \\ 85 & 10 \end{vmatrix}$$

$= 80(15 - 40) - 10(270 - 340) + 2(900 - 425)$

$= 80\,(-25) - 10\,(-70) + 2\,(475)$

$= -2000 + 700 + 950 = -350$

$$\Delta y = \begin{vmatrix} 9 & 80 & 2 \\ 13 & 90 & 4 \\ 6 & 85 & 3 \end{vmatrix} = 9\begin{vmatrix} 90 & 4 \\ 85 & 3 \end{vmatrix} - 80\begin{vmatrix} 13 & 4 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 13 & 90 \\ 6 & 85 \end{vmatrix}$$

$= 9(270 - 340) - 80(39 - 24) + 2(1105 - 540)$

$= 9(-70) - 80(15) + 2(565)$

$= -630 - 1200 + 1130 = -700$

$$\Delta z = \begin{vmatrix} 9 & 10 & 80 \\ 13 & 5 & 90 \\ 6 & 10 & 85 \end{vmatrix} = 9\begin{vmatrix} 5 & 90 \\ 10 & 85 \end{vmatrix} - 10\begin{vmatrix} 13 & 90 \\ 6 & 85 \end{vmatrix} + 80\begin{vmatrix} 13 & 5 \\ 6 & 10 \end{vmatrix}$$

$= 9(425 - 900) - 10(1105 - 540) + 80(130 - 30)$

$= 9(-475) - 10(565) + 80\,(100)$

$= -4275 - 5650 + 8000 = -1925$

$$x = \frac{\Delta x}{\Delta} = \frac{\overset{2}{\cancel{-350}}}{\cancel{-175}} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{4}{\cancel{-700}}}{\cancel{-175}} = 4$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{11}{\cancel{-1925}}}{\cancel{-175}} = 11$$

Thus the rates of commission on the items A, B and C are ₹ 2, ₹4 and ₹11 respectively.

10. The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 60% of those who already subscribe will subscribe again while 25% of those who do not now subscribe will subscribe. On the last letter it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

Sol : Let A represents the percent of people who subscribe the magazine and B represents the percent of people who do not subscribe the magazine.

Given 60% of people subscribe again implies 40% of people do not subscribe. And 25% of people are going to subscribe implies 75% of people are not going to subscribe.

$\therefore$ Transition probability matrix

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{pmatrix} \end{array}$$

Also, it is given that 40 % of those received the order of subscription implies 60% are not going to receive the order.

$\therefore (0.4 \quad 0.6)\begin{pmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{pmatrix}$

$= \big((0.4)(0.6)+(0.6)(0.25) \quad (0.4)(0.4)+(0.6)(0.75)\big)$

$= (0.24+0.15 \quad 0.16+0.45) = (0.39 \quad 0.61)$

$\therefore$ 39% of people who received the current letter can be expected to order a subscription.

PTA Questions & Answers

1 MARK

1. **If O(A) = 3 × 3 and ρ(A) = 2 then ρ(adjA) is________.** **[PTA-1]**

(a) 1 (b) 2 (c) 3 (d) 0

[Ans: (a) 1]

2. **If A is matrix [A,B] is the augmented matrix then which of the following is true?** **[PTA-2]**

(a) $\rho([A,B]) = \rho(A)$ (b) $\rho([A,B]) \geq \rho(A)$

(c) $\rho([A,B]) = \rho(A) > n$

(d) $\rho([A,B]) < \rho(A)$ **[Ans: (b) $\rho([A,B]) \geq \rho(A)$]**

3. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then the rank of AA^T is : **[PTA-4] [HY-2019]**

(a) 1 (b) 2 (c) 3 (d) 0

[Ans: (a) 1]

4. If A is matrix of order 4 and |A| = –2 then the value of |adj (A)| is ______ **[PTA-6]**

(a) –4 (b) 4 (c) –8 (d) 8

[Ans: (c) – 8**]**

Hint: $|\text{adj } A| = |A|^{n-1} = (-2)^{4-1} = (-2)^3 = -8$

2 MARKS

1. If $A = \begin{bmatrix} x & x & x \\ 4 & -2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ **find x if $\rho(A) = 3$** **[PTA - 3]**

Sol : $\begin{vmatrix} x & x & x \\ 4 & -2 & 1 \\ 2 & 3 & 4 \end{vmatrix} = x[-8-3]-x[16-2]+x[12+4]$

$= -11x - 14x + 16x$

$= -9x - 9x \neq 0$

$x \neq 0$

2. Find the rank of the matrix $\begin{bmatrix} 2 & -4 & x \\ 4 & -8 & 2x \\ 5 & 7 & 1 \end{bmatrix}$ **[PTA - 4]**

Sol : $\begin{bmatrix} 2 & -4 & x \\ 4 & -8 & 2x \\ 5 & 7 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{R_1}{2}} \begin{bmatrix} 1 & -2 & \frac{x}{2} \\ 4 & -8 & 2x \\ 5 & 7 & 1 \end{bmatrix}$

$\xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 4R1} \begin{bmatrix} 1 & -2 & \frac{x}{2} \\ 0 & 0 & 0 \\ 0 & 17 & \frac{2-5x}{2} \end{bmatrix}$

Rank 2 $\rho(A) = 2$

3. Parithi is either Sad (S) or happy (H) each day. If he is happy in one day, he is sad on the next day by four times out of five. If he is sad on one day, he his happy on the next day by two times out of three. Over a long run, what are the chances that Parithi is happy on any given day? **[PTA - 5; QY - 2019]**

Sol : The transition probability matrix is $T = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

At equilibrium, $(S\ H)\begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} = (S\ H)$

where $S + H = 1$

$\frac{4}{5}S + \frac{2}{3}H = S \Rightarrow \frac{4}{5}S + \frac{2}{3}(1-S) = S$

On solving this, we get

$S = \frac{10}{13}$ and $H = \frac{3}{13}$

In the long run, on a randomly selected day, his chances of being happy is $\frac{10}{13}$.

4. Find the rank of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix}$ **[PTA - 5]**

Sol : $\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 2[-4-2]-3[2-1]+4[-2-2]$

$= 2[-6]-3[1]+4[-4] = -12-3-16 = -31$

$\therefore \rho(A) = 3$

5. Six men and five women can jointly finish of work in 12 days, whereas five men and seven women can jointly finish the same work in 10 days, represent data as a system of linear equations. **[PTA - 6]**

Sol : One man finish work in x days

One woman finish work in y days.

Work done by one man in one day $\frac{1}{x}$

Work done by one woman in one day $\frac{1}{y}$

$\frac{6}{x} + \frac{5}{y} = \frac{1}{12} \Rightarrow \frac{5}{x} + \frac{7}{y} = \frac{1}{10}$

3 MARKS

1. Consider the matrix of transition probabilities of a product available in the market in two brands A and B. $\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$

Determine the market share of each brand in equilibrium position. **[PTA - 1]**

Sol : Transition probability matrix

$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$

At equilibrium, (A B) T=(AB) where A+B=1

$(A \quad B)\begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} = (A \quad B)$

$$\begin{aligned} 0.9A + 0.3B &= A \\ 0.9A+0.3(1-A) &= A \\ 0.9A-0.3A+0.3 &= A \\ 0.6A + 0.3 &= A \\ 0.4A &= 0.3 \\ A &= \frac{0.3}{0.4} = \frac{3}{4} \\ B &= 1-\frac{3}{4} = \frac{1}{4} \end{aligned}$$

Hence the market share of brand A is 75% and the market share of brand B is 25%

2. The total cost of 11 pencils and 3 erasers is ₹ 64 and the total cost of 8 pencils and 3 erasers is ₹49. Find the cost of each pencil and each eraser by Cramer's rule. **[PTA-2, March -2020]**

Sol : Let 'x' be the cost of a pencil

Let 'y' be the cost of an eraser

∴ By given data, we get the following equations

$11x + 3y = 64$

$8x + 3y = 49$

$$\Delta = \begin{vmatrix} 11 & 3 \\ 8 & 3 \end{vmatrix} = 9 \neq 0. \text{ It has unique solution.}$$

$$\Delta_x = \begin{vmatrix} 64 & 3 \\ 49 & 3 \end{vmatrix} = 45 \quad \Delta_y = \begin{vmatrix} 11 & 64 \\ 8 & 49 \end{vmatrix}$$

∴ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{45}{9} = 5$$

$$y = \frac{\Delta_y}{\Delta} = \frac{27}{9} = 3$$

∴ The cost of a pencil is ₹ 5 and the cost of an eraser is ₹ 3.

3. Using Cramer's rule, solve: $3e^x + 2e^y = 7$, $4e^x + 7e^y = 18$ **[PTA - 4]**

Sol : $3e^x + 2e^y = 7$; $4e^x + 7e^y = 18$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix} = 21 - 8 = 13$$

$$\Delta e^x = \begin{vmatrix} 7 & 2 \\ 18 & 7 \end{vmatrix} = 49 - 36 = 13$$

$$\Delta e^y = \begin{vmatrix} 3 & 7 \\ 4 & 18 \end{vmatrix} = 54 - 28 = 26$$

$$e^x = \frac{\Delta e^x}{\Delta} = \frac{13}{13} = 1 \Rightarrow e^y = \frac{\Delta e^y}{\Delta} = \frac{26}{13} = 2$$

$$e^x = 1 \log 1 = x = 0 \Rightarrow e^y = 2 \log 2 = y$$

5 MARKS

1. The price of 3 Business Mathematics books, 2 Accountancy books and one Commerce book is ₹840. The price of 2 Business Mathematics books, one Accountancy book and one Commerce book is ₹570. The price of one Business Mathematics book, one Accountancy book and 2 Commerce books is ₹630. Find the cost of each book by using Cramer's rule.

Sol : Let 'x' be the cost of a Business Mathematics book **[PTA - 1]**

Let 'y' be the cost of a Accountancy book.

Let 'z' be the cost of a Commerce book.

$$\begin{aligned} \therefore 3x + 2y + z &= 840 \\ 2x + y + z &= 570 \\ x + y + 2z &= 630 \end{aligned}$$

$$\text{Here, } \Delta = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2 \neq 0$$

$$\Delta x = \begin{vmatrix} 840 & 2 & 1 \\ 570 & 1 & 1 \\ 630 & 1 & 2 \end{vmatrix} = -240$$

$$\Delta y = \begin{vmatrix} 3 & 840 & 1 \\ 2 & 570 & 1 \\ 1 & 630 & 2 \end{vmatrix} = -300$$

$$\Delta z = \begin{vmatrix} 3 & 2 & 840 \\ 2 & 1 & 570 \\ 1 & 1 & 630 \end{vmatrix} = -360$$

∴ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{-240}{-2} = 120 \quad y = \frac{\Delta y}{\Delta} = \frac{-300}{-2} = 150$$

$$z = \frac{\Delta z}{\Delta} = \frac{-360}{-2} = 180$$

∴The cost of a Business Mathematics book is ₹120,

the cost of a Accountancy book is ₹150 and

the cost of a Commerce book is ₹180.

2. Solve by using rank method $x + y + 2z = 4$, $2x + 2y + 4z = 8$, $3x + 3y + 6z = 12$ **[PTA - 3]**

Sol : The matrix equation corresponding to the given

$$\text{system is} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

$$[A\ \ B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 2 & 4 & 8 \\ 3 & 3 & 6 & 12 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Back substituting we get

$x + y + 2z = 4$

$x = 4 - y - 2z$, where y and z are arbitrary

3. Investigate for what values of 'a' and 'b' the following system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + az = b$ have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

[PTA-3, March - 2020]

Sol : The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ b \end{pmatrix}$$

A X = B

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Case (i) For no solution:

The system possesses no solution only when $\rho(A) \neq \rho([A,B])$ which is possible only when $a-3=0$ and $b-10 \neq 0$

Hence for $a = 3$, $b \neq 10$, the system possesses no solution.

Case (ii) For a unique solution:

The system possesses a unique solution only when $\rho(A) = \rho([A,B])$ = number of unknowns.

i.e when $\rho(A) = \rho([A,B]) = 3$

Which is possible only when a−30 and b may be any real number as we can observe .

Hence for $a \neq 3$ and $b \in R$, the system possesses a unique solution.

Case (iii) For an infinite number of solutions:

The system possesses an infinite number of solutions only when

$\rho(A) = \rho([A,B]) <$ number of unknowns

i,e when $\rho(A) = \rho([A,B]) = 2 < 3$ (number of unknowns) which is possible only

when $a-3=0$, $b-10=0$

Hence for $a = 3$, $b = 10$, the system possesses infinite number of solutions.

4. Metro rail transit system has just gone into operation in a city. Of those who use the transit system this year 15% will switch over to using their own car next year and 85% will continue to use the transit system. Of those who use their cars this year, 70% will continue to use their cars next year and 30% will switch over to the transit system. Suppose the population of the city remains constant and that 60% of the commuters use the transit system and 40% of the commuters use their own car this year,

(i) What will be the change in commuter's usage after one year.

(ii) What percent of commuters will be using the Metro train system in the long run?

Sol : Transition probability matrix **[PTA - 4]**

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} .85 & .15 \\ .30 & .70 \end{pmatrix} \end{array}$$

Where A represent the percentage of people using metro rail transit system and B represents the percentage of people using car.

$A = 60\% = 0.60$ and $B = 40\% = 0.40$

$$\begin{pmatrix} 0.60 & 0.40 \end{pmatrix} \begin{pmatrix} 0.85 & 0.15 \\ 0.30 & 0.70 \end{pmatrix}$$

$= [(0.6)(0.85) + (0.4)(0.3)\ \ (0.6)(0.15) + (0.4)(0.7)]$

$= (0.63 \quad 0.37)$

$A = 63\% \quad B = 37\%$

(i) The percentage of commuters using the metro rail system after one year is 63% and the percent of commuters using the car after one year is 37%

(ii) Equilibrium will be reached in the long run. At equilibrium we must have

$$(A \quad B)T = (A \quad B)$$

Where $A + B = 1$

$$\Rightarrow (A \quad B)\begin{pmatrix} 0.85 & 0.15 \\ 0.30 & 0.70 \end{pmatrix} = (A \quad B)$$

$$(0.85A + 0.30\,B \;\; 0.15A + 0.70B) = (A \quad B)$$

$$\begin{aligned} 0.85A + 0.30B &= A \\ 0.85A + (0.3)(1 - A) &= A \\ 0.85A + 0.3 - 0.3A &= A \\ 0.3 &= A + 0.3A - 0.85\,A \\ 0.3 &= A(1 + .3 - 0.85) \\ 0.3 &= A\,(0.45) \\ A &= \frac{.3}{.45} = \frac{30}{45} = \frac{2}{3} \end{aligned}$$

The percent of commuters using the transit system in the long run is $66\frac{2}{3}\%$

5. Find x, y, z for the following system of equations **[PTA-5]**

$$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = 14, \; \frac{3}{x} - \frac{2}{y} + \frac{1}{z} = 3 \;\; \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 5$$

Sol : Put $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$

$$2a + 3b + 4c = 14 \quad ...(1)$$
$$3a - 2b + c = 3 \quad ...(2)$$
$$a + b + c = 5 \quad ...(3)$$

Matrix form of the system

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ 5 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 2 & 3 & 4 & 14 \\ 3 & -2 & 1 & 3 \\ 1 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R_1 \to R_3} \begin{bmatrix} 2 & 3 & 4 & 14 \\ 3 & -2 & 1 & 3 \\ 1 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & -5 & -2 & -12 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 5R_2} \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 8 & 8 \end{bmatrix}$$

Equivalent system

$$\begin{aligned} 8c &= 8 \\ \Rightarrow \quad c &= 1 \\ b + 2c &= 4 \\ \Rightarrow \quad b &= 4–2 = 2 \\ a + b + c &= 5 \\ \Rightarrow \quad a &= 5 - 2– 1 = 2 \\ \frac{1}{x} &= 2 \\ \Rightarrow \quad x &= \frac{1}{2} \\ \frac{1}{y} &= 2 \\ \Rightarrow \quad y &= \frac{1}{2} \\ \frac{1}{z} &= 1 \\ \Rightarrow \quad z &= 1 \end{aligned}$$

6. 80% of students who do maths work during one study period, will do the maths work at the next study period. 30% of students who do english work during one study period, will do the english work at the next study period. Initially there were 60 students do maths work and 40 students do english work. Calculate,

(i) The transition probability matrix **[PTA-6]**

(ii) The number of students who do maths work, english work for the next subsequent 2 study periods.

Sol : (i) Transition probability matrix

$$T = \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array}$$

After one study period,

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ (60 \quad 40) \end{array} \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} M & E \end{array} \\ (76 \quad 24) \end{array}$$

So in the very next study period, there will be 76 students do maths work and

24 students do the English work.
After two study periods,

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 76 & 24 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array} = (60.8 + 16.8 \quad 15.2+7.2)$$

$= 77.6 \quad 22.4$

After two study periods there will be 78 (approx) students do maths work and 22 (approx) students do English work.

Govt. Exam Questions & Answers

1 MARK

1. **The system of equations $2x - y = 1$, $3x + 2y = 12$ has** **[QY-2019]**
   (a) a unique solution
   (b) no solution
   (c) infinitely many solution
   (d) none of these **[*Ans.* (a) a unique solution]**

2. **If $|A| = 13$ and $|\text{Adj } A| = \begin{vmatrix} 4 & x \\ 5 & 7 \end{vmatrix}$, then the value of x is :** **[March - 2020]**
   (a) 3 (b) 4 (c) 2 (d) –5
   [*Ans.* (a) 3]

Hint: $[\because |\text{Adj } A| = |A|^{n-1}$ order $n \times n$

$$\therefore |A|^{2-1} = \begin{vmatrix} 4 & x \\ 5 & 7 \end{vmatrix}$$
$$|A| = 28 - 5x$$
$$5x = 28 - 13 = 15$$
$$5x = 15$$
$$x = \frac{15}{5} = 3$$

3. **The system has a unique solution when two lines :** **[Sep.-2020]**
   (a) L_1 and L_2 intersect exactly at one point
   (b) L_1 and L_2 coincides
   (c) L_1 and L_2 are parallel and distinct
   (d) Both (a) and (b) **[*Ans.* (d) Both (a) and (b)]**

2 MARKS

1. **Solve the equations $2x + 3y = 7$, $3x + 5y = 9$ by using Cramer's rule.** **[Govt. MQP - 2019]**

Sol : $\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1 \neq 0.$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 7 & 3 \\ 9 & 5 \end{vmatrix} = 7(5) - 9\,(3)$$
$$= 35 - 27 = 8$$
$$\Delta y = \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} = 2(9) - 3\,(7)$$
$$= 18 - 21 = -3$$
$$\therefore x = \frac{\Delta x}{\Delta} = \frac{8}{1} = 8$$
$$y = \frac{\Delta y}{\Delta} = \frac{-3}{1} = -3$$

$\therefore$ Solution set is $\{8, -3)$

2. **If the rank of the matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & K \\ 9 & 10 & 11 & 12 \end{bmatrix}$ is 2. Find the value of 'K'** **[HY - 2019]**

Sol : $\begin{vmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & k \end{vmatrix} = 0 \Rightarrow k = 7$

3. **Find the rank of the matrix A = $\begin{vmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{vmatrix}$** **[Sep. - 2020]**

Sol : The order of A is 3×4

$\therefore \rho(A) \leq 3$

Let us transform the matrix A to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \to R_3 - 5R_2$

The number of non zero rows is 3. $\therefore \rho(A) = 3$.

3 MARKS

1. **Find the rank of the matrix**

$$A = \begin{pmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{pmatrix}$$ **[Govt. MQP - 2019]**

Sol : The order of A is 3×4

$$\therefore \rho(A) \leq \min(3, 4)$$
$$\rho(A) \leq 3$$

Consider the third order minor,

$$\begin{vmatrix} 1 & 2 & -4 \\ 2 & -1 & 3 \\ 8 & 1 & 9 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & -1 \\ 8 & 1 \end{vmatrix}$$

$= 1(-9-3) - 2(18-24) - 4(2+8)$

$= 1(-12) - 2(-6) - 4(10)$

$= -12 + 12 - 40 = -40 \neq 0.$

There is a minor of order 3, which is not zero

$\therefore \quad \rho(A) = 3.$

2. **Show that the equations $x + y = 5$, $2x + y = 8$ are consistent and solve them.** **[QY - 2019]**

Sol : The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

Matrix A	Augment matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 1 & 8 \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	$\sim\begin{pmatrix} 1 & 1 & 5 \\ 1 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A) = 2$	$\rho([A,B]) = 2$	

Number of non-zero rows is 2.

$\rho(A) = \rho([A,B]) = 2 =$ Number of unknowns.

The given system is consistent and has unique solution.

Now, the given system is transformed into

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$x + y = 5$$
$$y = 2$$
$$\therefore (1)\; x + 2 = 5$$
$$x = 3$$

Solution is $x = 3, y = 2$

3. **Show that the equations $x - 4y + 7z = 14$, $3x + 8y - 2z = 13$, $7x - 8y + 26z = 5$ are inconsistent.** **[HY - 2019]**

Sol : $[A, B] = \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{array}\right]$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & 64 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \\ R_3 \to R_3 - R_2 \end{array}$$

$\to \rho(A) = 2, \rho(A, B) = 3$

$\Rightarrow \rho(A) \neq \rho(A, B)$

The system is inconsistent and had no solution.

4. **If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B^T = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$, then find the rank of AB.** **[March - 2020]**

Sol : $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2-5 & -2+4-1 & 3-6+1 \\ 2+6+20 & -4-12+4 & 6+18-4 \\ 3+4+15 & -6-8+3 & 9+12-3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix} = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$$

Matrix (AB)	Elementary Transformation
$AB = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -6 & -2 \\ -12 & 28 & 20 \\ -11 & 22 & 18 \end{pmatrix}$	$C_1 \leftrightarrow C_2$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ -11 & 22 & 18 \end{pmatrix}$	$R_2 \to R_2 + 12R_1$

Matrix (AB)	Elementary Transformation
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & -44 & -4 \end{pmatrix}$	$R_3 \to R_3 + 11R_1$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The matrix is in echelon form and the number of non-zero rows is 2.

$\therefore \rho(AB) = 2.$

5 MARKS

1. The values of a quadratic polynomial of $f(x) = ax^2 + bx + c$ are given by $f(1) = 0$; $f(2) = -2$ and $f(3) = -6$. Using Cramer's rule find the value of a, b and c. **[Sep. - 2020]**

Sol : Let $f(x) = ax^2 + bx + c$

$$\begin{aligned}
\text{Given } f(1) &= 0, f(2) = -2 \text{ and } f(3) = -6 \\
f(1) &= 0 \\
\Rightarrow \quad 0 &= a(1) + b(1) + c \\
\Rightarrow \quad a + b + c &= 0 \quad \ldots (1) \\
f(2) &= -2 \\
\Rightarrow \quad -2 &= a(2^2) + b(2) + c \\
\Rightarrow \quad -2 &= 4a + 2b + c \\
\Rightarrow 4a + 2b + c &= -2 \quad \ldots (2) \\
f(3) &= -6 \\
\Rightarrow \quad -6 &= a(3^2) + b(3) + c \\
\Rightarrow \quad -6 &= 9a + 3b + c \\
\Rightarrow 9a + 3b + c &= -6 \quad \ldots (3)
\end{aligned}$$

$$\therefore \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 9 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 9 & 3 \end{vmatrix}$$

$$= 1(2-3) - 1(4-9) + 1(12-18)$$

$$= -1 + 5 - 6 = -2$$

Since $\Delta \neq 0$, Cramer's rule is applicable

$$\Delta_a = \begin{vmatrix} 0 & 1 & 1 \\ -2 & 2 & 1 \\ -6 & 3 & 1 \end{vmatrix} = 0 - 1\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & 2 \\ -6 & 3 \end{vmatrix}$$

$$= -1(-2+6) + 1(-6+12)$$

$$= -4 + 6 = 2$$

$$\Delta_b = \begin{vmatrix} 1 & 0 & 1 \\ 4 & -2 & 1 \\ 9 & -6 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix} - 0 + 1\begin{vmatrix} 4 & -2 \\ 9 & -6 \end{vmatrix}$$

$$= 1(-2+6) - 0 + 1(-24+18)$$

$$= 4 - 6 = -2$$

$$\Delta_c = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & -2 \\ 9 & 3 & -6 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ 3 & -6 \end{vmatrix} - 1\begin{vmatrix} 4 & -2 \\ 9 & -6 \end{vmatrix} + 0$$

$$= 1(-12+6) - 1(-24+18) = -6 + 6 = 0$$

By Cramer's rule,

$$a = \frac{\Delta_a}{\Delta} = \frac{2}{-2} = -1$$

$$b = \frac{\Delta_b}{b} = \frac{-2}{-2} = -1$$

$$c = \frac{\Delta_c}{c} = \frac{0}{-2} = 0$$

$$\therefore a = -1, b = 1, c = 0$$

Additional Questions & Answers

1 MARK

I. Choose the correct answer :

1. If the minor of a_{23} = the co-factor of a_{23} in $|a_{ij}|$ then the minor of a_{23} is.

(a) 1 (b) 2 (c) 0 (d) 3

[Ans: (c) 0]

2. If AB = BA = |A| I then the matrix B is the.

(a) inverse of A (b) Transpose of A

(c) Adjoint of A (d) 2A

[Ans: (a) inverse of A]

3. If A is a square matrix of order 3, then |adj A| is

(a) $|A|^2$ (b) $|A|$ (c) $|A|^3$ (d) $|A|^4$

[Ans: (a) $|A|^2$]

4. If |A| = 0, then |adj A| is.

(a) 0 (b) 1 (c) –1 (d) ± 1

[Ans: (a) 0]

5. For what value of k, the matrix A = $\begin{pmatrix} 2 & k \\ 3 & 5 \end{pmatrix}$ has no inverse?

(a) $\frac{3}{10}$ (b) $\frac{10}{3}$ (c) 3 (d) 10

[Ans: (b) $\frac{10}{3}$]

6. **The rank of an $n \times n$ matrix each of whose elements is 2 is**

(a) 1 (b) 2 (c) n (d) n^2

[Ans: (a) 1]

7. **The value of** $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ **is.**

(a) 5^2 (b) 0 (c) 5^{13} (d) 5^9

[Ans: (b) 0]

8. **If** $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$ **then x =**

(a) 3 (b) ± 3 (c) ± 6 (d) 6

[Ans: (c) $\pm$ 6]

9. **If A is a singular matrix, then Adj A is.**

(a) non-singular (b) singular
(c) symmetric (d) not defined

[Ans: (b) singular]

10. **If A, B are two $n \times n$ non-singular matrices, then.**

(a) AB is non-singular (b) AB is singular
(c) $(AB)^{-1} = A^{-1} B^{-1}$
(d) $(AB)^{-1}$ does not exit

[Ans: (a) AB is non-singular]

II. Fill in the blanks :

1. **If A =** $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ **then the value of |adj A| is ______** **[Ans: a^6]**

2. **For any 2 × 2 matrix, if A (adj A) =** $\begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}$ **then |A| is ________.**

[Ans: 10]

3. **If A is a square matrix of order n, then |Adj A| = ______.** **[Ans: $|A|^{n-1}$]**

4. **If A is a matrix of order 3 and |A| = 8 then |adj A| = _______.** **[Ans: 64]**

5. **If A is a square matrix such that $A^2 = I$, then A^{-1} = ________.** **[Ans: A]**

6. **The system of equation $x + y + z = 2$, $3x - y + 2z = 6$ and $3x + y - z = -18$ has ________ solution** **[Ans: unique]**

7. **The number of solutions of the system of equations $2x + y - z = 7$, $x - 3y + 2z = 1$, $x + 4y - 3z = 5$ is _____.** **[Ans: 0]**

8. **The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if k is________.** **[Ans: Not equal to 0]**

9. **The value of λ for which the system of equations x + y + z = 5, $x + 2y + 3z = 9$, $x + 3y + \lambda z = \mu$ is ______.** **[Ans: $\lambda \neq 5$]**

10. **A set of values of the variable x_1, x_2,....x_n satisfying all the equations simultaneously is called ________ of the system.** **[Ans: Solution]**

III. Match the following :

1.	Rank of a matrix	i.	1
2.	If A is a matrix of order $m \times n$, then $\rho(A) \leq$	ii.	non-zero row
3.	Rank of a zero matrix is	iii.	n
4.	Rank of a non-singular matrix of order $n \times n$ is	iv.	unique solution
5.	If A is of rank 2, then adj A is of rank	v.	inconsistent
6.	A row having at least one non-zero element is	vi.	infinitely many solutions
7.	For the system of equations AX = B, the solution is $X = A^{-1}B$ provided	vii.	$\leq \min\{m, n\}$
8.	If $\rho(A, B) = \rho(A) < n$ then the system has	viii.	0
9.	If $\rho(A, B) = \rho(A) = n$, then the system has	ix.	$\|A\| \neq 0$
10.	If $\rho(A, B) \neq \rho(A)$ then the system is	x.	≥ 0

[Ans: 1 - x, 2 - vii, 3 - viii, 4 - iii, 5 - i, 6 - ii, 7 - ix, 8 - vi, 9 - iv, 10 - v]

IV. Choose the odd one out:

1. **The system of non-homogeneous equations will have.**

(a) unique solution
(b) Infinitely many solutions
(c) No solution (d) Trivial solution

[Ans: (d) Trivial solution]

2. **Rank of a 2 × 2 matrix may be**

(a) 0 (b) 1 (c) 2 (d) 3

[Ans: (d) 3]

3. The transition probabilities P_{jk} satisfy

(a) $P_{jk} > 0$ (b) $\sum_{k}^{1} P_{jk} = 1$ for all j

(c) $P_{jk} \leq 0$ (d) $P_{jk} > 1$

[Ans: (c) $P_{jk} \leq 0$]

4. If |A| = 0, then

(a) A is a singular matrix

(b) System has either no solution or infinitely many solutions

(c) No solution

(d) non-singular matrix

[Ans: (d) non-singular matrix]

V. Which is the following is not correct in the given statement:

1. (a) $|A| = |A^T|$ where $A = [a_{ij}]_{3 \times 3}$

(b) $|KA| = K^3 |A|$ where $A = [a_{ij}]_{3 \times 3}$

(c) If is a non-singular matrix, then $|A| \neq 0$

(d) $\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$

[Ans: (d) $\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$]

2. For the matrix $A = [a_{ij}]_{3 \times 3}$

(a) Order of minor is less than the order of |A|.

(b) Minor of an element can never be equal to co-factor of the same element.

(c) Value of a determinant is obtained by multiplying elements of a row or column by corresponding factors

(d) Order of minor and co-factors of elements of A is same.

[Ans: (b) Minor of an element can never be equal to co-factor of the same element.]

3. If A is an invertible matrix, then which of the following is not true?

(a) $(A^2)^{-1} = (A^{-1})^2$ (b) $|A^{-1}| = |A|^{-1}$

(c) $(A^T)^{-1} = (A^{-1})^T$ (d) $|A| \neq 0$

[Ans: (a) $(A^2)^{-1} = (A^{-1})^2$]

4. (a) If three planes intersect at a point, then the system has unique solution.

(b) If three planes intersect along a line then the system has infinitely many solutions lying on this line.

(c) If two planes intersect at a point then the system has unique solution.

(d) If three planes are parallel and distinct and there is no point in common, then the system has no solutions

[Ans: (c) If two planes intersect at a point then the system has unique solution]

5. Solution of the system of equations $x + 2y = 7$ and $3x + 6y = 21$ is

(a) $x = 5, y = 1$ (b) $x = 3, y = 2$

(c) $x = 0, y = 1$ (d) $x = -3, y = 5$

[Ans: (c) $x = 0, y = 1$]

2 MARKS

1. Find the rank of the matrix $\begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix}$

Sol : Let A $= \begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix}$

The order of A is 2 × 2

$\rho(A) \leq \min(2, 2)$

$\Rightarrow \rho(A) \leq 2.$

$\begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix} = 7-(-2) = 7+2 = 9 \neq 0$

The highest order of non-vanishing minor of A is 2

$\therefore \rho(A) = 2$

2. Find the rank of the matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

Sol : Let A $= \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

The order of A is 2 × 2

$\rho(A) \leq \min(2, 2)$

$\Rightarrow \rho(A) \leq 2.$

$\begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 4-4 = 0.$

Since the second order minor vanishes, $\rho(A) \neq 2$.

We have to try for atleast one non-zero first order minor.

i.e. atleast one non-zero element of A.

This is possible because A has non-zero element.

$\therefore \rho(A) = 1.$

3. Solve $x + 2y = 3$ and $x + y = 2$ using Cramer's rule.

Sol : $\Delta = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = (1)(1) - (1)(2) = 1 - 2 = -1$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 3 - 4 = -1$$

$$\Delta y = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1$$

$$x = \frac{\Delta x}{\Delta} = \frac{-1}{-1} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-1}{-1} = 1$$

$\therefore$ solution set is $\{1, 1\}$

4. Solve: $x + 2y = 3$ and $2x + 4y = 6$ using rank method.

Sol : The non-homogeneous equations are

$x + 2y = 3$, $\quad 2x + 4y = 6$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$

Here $\rho(A) = 1$ and $\rho([A, B]) = 1$

Since $\rho(A) = \rho([A, B]) = 1 <$ Number of unknowns, the given system is consistent with infinitely many solutions.

To find the solution, let us rewrite the above echelon form into the matrix form, we get

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x + 2y = 3 \quad \ldots (1)$

let $y = k, \quad k \in R$

$(1) \Rightarrow \quad x + 2k = 3 \quad \Rightarrow \quad x = 3 - 2k$

$\therefore$ Solution set is $\{3 - 2k, k\}, k \in R$.

For different values of k, we get infinite number of solutions.

5. Show that the equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $x + 4y + 7z = 30$ are consistent.

Sol : Given non-homogeneous equations are

$x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$.

Augmented matrix	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

Here $\rho(A) = 2$ and $\rho(A, B) = 2$

$\therefore \quad \rho(A) = \rho(A, B) = 2 <$ Number of unknowns.

$\therefore \quad$ The given system is consistent.

6. Solve : $2x + 3y = 4$ and $4x + 6y = 8$ using Cramer's rule.

Sol :

$$\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

$$\Delta x = \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 24 - 24 = 0$$

$$\Delta y = \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 16 - 16 = 0$$

$$\therefore \Delta = \Delta x = \Delta y = 0$$

$\therefore$ The system is consistent with infinite number of solutions.

let $y = k, \quad k \in R$

$\therefore 2x + 3k = 4 \Rightarrow 2x = 4 - 3k$

$\Rightarrow x = \frac{1}{2}(4 - 3k), k \in R$

$\therefore$ Solution set is $\left\{\frac{4-3k}{2}, K\right\}, k \in R$.

7. If A and B are non-singular matrices, prove that AB is non-singular.

Sol : Since A and B are non-singular,

$|A| \neq 0, \quad |B| \neq 0$

Consider $|AB| = |A| \cdot |B|$

$\neq 0$ since $|A| \neq 0$ and $|B| \neq 0$.

$\Rightarrow \quad |AB| \neq 0$

$\therefore$ AB is non-singular.

8. For what value of x, the matrix $A = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix}$ is singular?

Sol : The matrix A is singular, if $\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$

$$1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix} = 0$$

$\Rightarrow 1(-6-2) + 2(-3-x) + 3(2-2x) = 0$

$\Rightarrow 1(-8) - 6 - 2x + 6 - 6x = 0$

$\Rightarrow -8 - 2x - 6x = 0$

$\Rightarrow -8 - 8x = 0$

$\Rightarrow -8 = 8x$

$\Rightarrow x = \frac{-8}{8} = -1$

9. If $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ find x, y and z

Sol : Given $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x+0+0 \\ 0+y+0 \\ 0+0+z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\Rightarrow x = 1, \quad y = -1, \quad z = 0$

10. Two newspapers A and B are published in a city. Their market shares are 15% for A and 85% for B of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after one year.

Sol : Transition probability matrix $T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{pmatrix} \end{array}$

Given present market shares are 15% for A and 85% for B

$\therefore$ Market shares after one year

$= \begin{pmatrix} 0.15 & 0.85 \end{pmatrix}\begin{pmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{pmatrix}$

$= \begin{pmatrix} (0.15)(0.65)+(0.85)(0.45) & 0.15\times0.35+0.85\times0.55 \end{pmatrix}$

$= \begin{pmatrix} 0.0975+0.3825 & 0.0525+0.4675 \end{pmatrix}$

$= \begin{pmatrix} 0.48 & 0.52 \end{pmatrix}$

$\therefore$ Market shares after one year for A is 48% and for B is 52%

3 MARKS

1. Find the rank of the matrix $A = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{pmatrix}$

Sol : The order of A is 3×3

$\therefore \quad \rho(A) \leq \min(3, 3)$

$\Rightarrow \quad \rho(A) \leq 3$

Matrix	Elementary Transformation
$\begin{pmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{pmatrix}$	$C_1 \to C_1 \div 2$ $C_2 \to C_2 \div 4$ and $C_3 \to C_3 \div 5$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 + 3R_1$

The last equivalent matrix is in echelon form and it has one non-zero row.

$\therefore \rho(A) = 1$

2. Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{pmatrix}$

Sol : The order of A is 3×4

$\therefore \quad \rho(A) \leq \min(3, 4)$

$\rho(A) \leq 3$

Consider the third order minor,

$$\begin{vmatrix} 1 & 2 & -4 \\ 2 & -1 & 3 \\ 8 & 1 & 9 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & -1 \\ 8 & 1 \end{vmatrix}$$

$= 1(-9-3) - 2(18-24) - 4(2+8)$

$= 1(-12) - 2(-6) - 4(10)$

$= -12 + 12 - 40 = -40 \neq 0.$

There is a minor of order 3, which is not zero

$\therefore \rho(A) = 3$

3. Show that the equations $2x - y + z = 7$, $3x + y - 5z = 13$, $x + y + z = 5$ are consistent and have a unique solution.

Sol : The non-homogeneous equation are

$2x - y + z = 7,\ 3x + y - 5z = 13,\ x + y + z = 5$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 2 & -1 & 1 & 7 \\ 3 & 1 & -5 & 13 \\ 1 & 1 & 1 & 5 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 5 \\ 3 & 1 & -5 & 13 \\ 2 & -1 & 1 & 7 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -8 & -2 \\ 0 & -3 & -1 & -3 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -8 & -2 \\ 0 & 0 & 11 & 0 \end{pmatrix}$	$R_3 \to R_3 - \frac{3}{2} R_2$

Clearly $\rho(A) = 3$ and $\rho(A, B) = 3$

$\Rightarrow$ $\rho(A) = \rho(A, B) = 3$ = Number of unknowns

$\therefore$ The given system is consistent and has unique solution.

4. Show that the equations $x + 2y = 3, y - z = 2, x + y + z = 1$ are consistent and have infinite sets of solution.

Sol : Given non-homogeneous equations are

$x + 2y = 3, \qquad y - z = 2, \qquad x + y + z = 1$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{pmatrix}$	$R_3 \to R_3 + R_1$
$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 + R_2$

Obviously, $\rho(A) = 2$ and $\rho(A, B) = 2$

Hence $\rho(A) = \rho(A, B) = 2 <$ Number of unknowns.

$\therefore$ The system is consistent and has infinite number of solutions.

5. Show that the equations $x - 3y + 4z = 3$, $2x - 5y + 7z = 6$, $3x - 8y + 11z = 1$ are inconsistent.

Sol : Given non-homogeneous equations are

$x - 3y + 4z = 3, 2x - 5y + 7z = 6, 3x - 8y + 11z = 1$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & -3 & 4 & 3 \\ 2 & -5 & 7 & 6 \\ 3 & -8 & 11 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -8 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -0 & -8 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Clearly $\rho(A) = 2$ and $\rho(A, B) = 3$

$\rho(A, B) \neq \rho(A)$

Hence, the given system is inconsistent and has no solution.

6. Solve : $2x - 3y - 1 = 0, 5x + 2y - 12 = 0$ by Cramer's rule.

Sol : The non-homogeneous equations are

$2x - 3y - 1 = 0, \quad 5x + 2y - 12 = 0$

$$\Delta = \begin{vmatrix} 2 & -3 \\ 5 & 2 \end{vmatrix} = 4 + 15 = 19 \neq 0$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 1 & -3 \\ 12 & 2 \end{vmatrix} = 2 + 36 = 38$$

$$\Delta y = \begin{vmatrix} 2 & 1 \\ 5 & 12 \end{vmatrix} = 24 - 5 = 19$$

$$x = \frac{\Delta x}{\Delta} = \frac{38}{19} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{19}{19} = 1$$

$\therefore$ Solution set is $\{2, 1\}$

7. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ find x, y and z.

Sol : Given $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x+0+0 \\ 0+0+z \\ 0+y+0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$\Rightarrow x = 2, z = -1$ and $y = 3$

$\therefore$ Solution set is $\{2, 3, -1\}$

8. If A = $\begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$, X = $\begin{pmatrix} n \\ 1 \end{pmatrix}$ B = $\begin{pmatrix} 8 \\ 11 \end{pmatrix}$ and AX = B then find *n*.

Sol : Given AX = B

$$\begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 2n+4 \\ 4n+3 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

Equating the corresponding entries on both sides, we get

$$2n+4 = 8$$
$$2n = 8-4$$
$$2n = 4$$
$$n = \frac{4}{2}$$
$$n = 2.$$

9. Solve: $2x + 3y = 5$, $6x + 5y = 11$

Sol : Given non-homogeneous equations are

$2x + 3y = 5$, $6x + 5y = 11$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix} = 10 - 18 = -8$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 5 & 3 \\ 11 & 5 \end{vmatrix} = 25 - 33 = -8$$

$$\Delta y = \begin{vmatrix} 2 & 5 \\ 6 & 11 \end{vmatrix} = 22 - 30 = -8$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{-8}{-8} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-8}{-8} = 1$$

$\therefore$ Solution set is $\{1, 1\}$

10. Two products A and B currently share the market with shares 60% and 40% each respectively. Each week some brand switching latees place. Of those who bought A the previous week 70% buy it again whereas 30% switch over to B. Of those who bought B the previous week, 80% buy it again whereas 20% switch over to A. Find their shares after one week and after two weeks.

Sol : Transition probability matrix

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0\cdot7 & 0\cdot3 \\ 0\cdot2 & 0\cdot8 \end{pmatrix} \end{array}$$

Shares after one week $\begin{pmatrix} 0.6 & 0.4 \end{pmatrix}\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$

$= \begin{pmatrix} 0.6\times0.7+0.4\times0.2 & 0.6\times0.3+0.4\times0.8 \end{pmatrix}$

$= \begin{pmatrix} 0.42+0.08 & 0.18+0.32 \end{pmatrix} = \begin{pmatrix} 0.50 & 0.50 \end{pmatrix}$

$\Rightarrow$ A = 50% and B = 50%

Shares after two weeks $\begin{pmatrix} 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$

$= \begin{pmatrix} 0.5\times0.7+0.5\times0.2 & 0.5\times0.3+0.5\times0.8 \end{pmatrix}$

$= \begin{pmatrix} 0.35+0.10 & 0.15+0.40 \end{pmatrix} = \begin{pmatrix} 0.45 & 0.55 \end{pmatrix}$

A = 45% and B = 55%

5 MARKS

1. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result we get 7. By adding second and third numbers to three times the first number we get 12. Find the numbers using rank method.

Sol : Let the three numbers be x, y and z respectively

Given
$$x + y + z = 6$$
$$x + 2z = 7$$
$$3x + y + z = 12$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 0 & 2 & 7 \\ 3 & 1 & 1 & 12 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & -2 & -6 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -4 & -8 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The last equivalent matrix is in echelon form

$\rho(A) = 3$ and $\rho(A, B) = 3$

$\therefore$ $\rho(A) = \rho(A, B) = 3$ = Number of unknowns

$\therefore$ The system is consistent and has unique solution.

To find the solutions, let us rewrite the echelon form into matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$$

$$x + y + z = 6 \qquad \ldots (1)$$
$$-y + z = 1 \qquad \ldots (2)$$
$$-4z = -8 \qquad \ldots (3)$$

From (3), $-4z = -8 \Rightarrow z = \frac{-8}{-4} = 2$

Substituting $z = 2$ in (2) we get

$$-y + 2 = 1 \Rightarrow -y = 1 - 2 \Rightarrow -y = -1$$

$$\Rightarrow \quad y = 1$$

Substituting $y = 1$ and $z = 2$ in (1) we get,

$$x + 1 + 2 = 6 \Rightarrow x + 3 = 6 \Rightarrow x = 6 - 3$$

$$\Rightarrow \quad x = 3$$

Hence, the numbers are (3, 1, 2).

2. A mixture is to be made of three foods A, B, C. The three foods A, B, C contain nutrients P, Q, R as shown below.

Ounces per pound of Nutrient			
Food	**P**	**Q**	**R**
A	1	2	5
B	3	1	1
C	4	2	1

How to form a mixture which will have 8 ounces of P, 5 ounces of Q and 7 ounces of R? (Cramer's rule).

Sol : Let x pounds of food A, y pounds of food B and z pounds of food C be needed to form the mixture.

Given
$$x + 3y + 4z = 8$$
$$2x + y + 2z = 5$$
$$5x + y + z = 7$$

$$\Delta = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix}$$

$$= 1(1-2) - 3(2-10) + 4(2-5)$$

$$= 1(-1) - 3(-8) + 4(-3)$$

$$= -1 + 24 - 12 = 11$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system has unique solution.

$$\Delta x = \begin{vmatrix} 8 & 3 & 4 \\ 5 & 1 & 2 \\ 7 & 1 & 1 \end{vmatrix}$$

$$= 8\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} + 4\begin{vmatrix} 5 & 1 \\ 7 & 1 \end{vmatrix}$$

$$= 8(1-2) - 3(5-14) + 4(5-7)$$

$$= 8(-1) - 3(-9) + 4(-2)$$

$$= -8 + 27 - 8 = 11$$

$$\Delta y = \begin{vmatrix} 1 & 8 & 4 \\ 2 & 5 & 2 \\ 5 & 7 & 1 \end{vmatrix} = 1\begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} - 8\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix}$$

$$= 1(5-14) - 8(2-10) + 4(14-25)$$

$$= 1(-9) - 8(-8) + 4(-11)$$

$$= -9 + 64 - 44 = 11$$

$$\Delta z = \begin{vmatrix} 1 & 3 & 8 \\ 2 & 1 & 5 \\ 5 & 1 & 7 \end{vmatrix} = 1\begin{vmatrix} 1 & 5 \\ 1 & 7 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix}$$

$$= 1(7-5) - 3(14-25) + 8(2-5)$$

$$= 1(2) - 3(-11) + 8(-3)$$

$$= 2 + 33 - 24 = 11$$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{11}{11} = 1$$

$$\Rightarrow \quad y = \frac{\Delta y}{\Delta} = \frac{11}{11} = 1$$

$$z = \frac{\Delta z}{\Delta} = \frac{11}{11} = 1$$

Hence, the mixture is formed by mixing one pound of each of the foods A, B and C.

3. For what values of k, the system of equations $kx + y + z = 1, x + ky + z = 1, x + y + kz = 1$ have

(i) Unique solution

(ii) More than one solution

(iii) No solution

Sol : The given non-homogeneous equations can be written as

$$\begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim\begin{pmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - kR_1$
$\sim\begin{pmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & 1-k \end{pmatrix}$	$R_3 \to R_3 + R_2$
$\sim\begin{pmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & k^2+k-2 & k-1 \end{pmatrix}$	$R_3 \to R_3(-1)$
$\sim\begin{pmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & (k-1)(k+2) & k-1 \end{pmatrix}$	

Case (i) When $k \neq 1$, and $k \neq -2$

$\rho(A) = \rho(A, B) = 3$ = Number of unknowns.

$\therefore$ The system has unique solution.

Case (ii)

When $k = 1$ $\quad [A, B] \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\rho(A) = \rho(A, B) = 1 <$ Number of unknowns.

$\therefore$ The system is consistent and has infinitely many solutions.

Case (iii) When $k = -2$

$[A, B] \sim \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$

$\rho(A) = 2$, $\rho(A, B) = 3$

$\Rightarrow \rho(A) \neq \rho(A, B)$

$\therefore$ The system is inconsistent and has no solution.

4. Using determinants, find the quadratic defined by $f(x) = ax^2 + bx + c$ if $f(1) = 0$, $f(2) = -2$ and $f(3) = -6$.

Sol : Given $f(x) = ax^2 + bx + c$

$$f(1) = 0$$
$$\Rightarrow a(1)^2 + b(1) + c = 0$$
$$\Rightarrow a + b + c = 0 \quad ...(1)$$
$$f(2) = -2$$
$$\Rightarrow a(2^2) + b(2) + c = -2$$
$$\Rightarrow 4a + 2b + c = -2 \quad ...(2)$$
$$f(3) = -6$$
$$\Rightarrow a(3^2) + b(3) + c = -6$$
$$\Rightarrow 9a + 3b + c = -6 \quad ...(3)$$

Now $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix}$

$$= 1(2-3) - 1(4-9) + 1(12-18)$$
$$= -1 + 5 - 6 = -2 \neq 0.$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system has unique solution.

$$\Delta a = \begin{vmatrix} 0 & 1 & 1 \\ -2 & 2 & 1 \\ -6 & 3 & 1 \end{vmatrix}$$
$$= 0 - 1(-2+6) + 1(-6+12)$$
$$= -4 + 6 = 2$$
$$\Delta b = \begin{vmatrix} 1 & 0 & 1 \\ 4 & -2 & 1 \\ 9 & -6 & 1 \end{vmatrix}$$
$$= 1(-2+6) + 0 + 1(-24+18)$$
$$= 4 - 6 = -2$$
$$\Delta c = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & -2 \\ 9 & 3 & -6 \end{vmatrix}$$
$$= 1(-12+6) - 1(-24+18) + 0$$
$$= -6 + 6 = 0$$
$$a = \frac{\Delta a}{\Delta} = \frac{\cancel{2}}{\cancel{-2}} = -1$$
$$\Rightarrow b = \frac{\Delta b}{\Delta} = \frac{\cancel{-2}}{\cancel{-2}} = 1$$

$$c = \frac{\Delta c}{\Delta} = \frac{0}{-2} = 0$$

$$f(n) = (-1)x^2 + 1(x) + 0$$

$$\Rightarrow \quad f(x) = -x^2 + x.$$

5. A new transit system has just gone into operation in a city. Of those who use the transit system this year, 10% will switch over to using their own car next year and 90% will continue to use the transit system. Of those who use their cars this year, 80% will continue to use their cars next year and 20% will switch over to the transit system. Suppose the population of the city remains constant and that 50% of the commuters use the transit system and 50% of the commuters use their own car this year,

(i) What percent of commuters will be using the transit system after one year?

(ii) What percent of commuters will be using the transit system in the long run?

Sol : Let A represents the percent of commuters who use the transit system and B represents the percent of commuters who use their own car.

Transition probability matrix $T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$

Given 50% of commuters use the transit system and 50% of the commuters use their own car this year.

(i) Percentage of commuters after one year

$$\begin{pmatrix} 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5\times0.9+0.5\times0.2 & 0.5\times0.1+0.5\times0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.45+0.10 & 0.05+0.40 \end{pmatrix} = \begin{pmatrix} 0.55 & 0.45 \end{pmatrix}$$

A = 55% and B = 45%

(ii) Equilibrium will be reached in the long run at equilibrium, we must have

$$\begin{pmatrix} A & B \end{pmatrix} T = \begin{pmatrix} A & B \end{pmatrix}$$

where $A + B = 1$

$$\Rightarrow \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

$$\begin{pmatrix} 0.9A + 0.2B & 0.1A + 0.8B \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

Equating the corresponding entries on both sides we get,

$$0.9A + 0.2B = A$$

$$\Rightarrow 0.9A + 0.2(1 - A) = A$$

[Since $A + B = 1 \Rightarrow B = 1 - A$]

$$\Rightarrow 0.9A + 0.2 - 0.2A = A$$

$$\Rightarrow \quad 0.2 = A - 0.9A + 0.2A$$

$$\Rightarrow \quad 0.2 = A(1 - 0.9 + 0.2)$$

$$\Rightarrow \quad 0.2 = A(0.3)$$

$$\Rightarrow \quad A = \frac{0.2}{0.3} = 0.666$$

$$\Rightarrow \quad A = 67\%$$

$\therefore$ 67% of the commuters will be using the transit system in the long run.

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