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Strictly as per the Reduced (Prioritised) Syllabus released on 13th August, 2021 (G.O.(Ms).No126)

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PREFACE

Sir,
An equation has no meaning, for me
unless it expresses a thought of GOD

- Ramanujam
[Statement to a friend]

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers,

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **Sura's Mathematics Guide Volume I and Volume II** for +2 Standard. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises and in-text questions in addition to precise answers for textual questions.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

Subash Raj, B.E., M.S.
- Publisher
Sura Publications

All the Best

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Strictly as per the Reduced (Prioritised) Syllabus released on 13th August, 2021 (G.O.(Ms).No126)

UNIT	CONTENT
4.2.4	Inverse functions
4.2.5	Graphs of inverse functions
4.3	Sine Function and Inverse Sine Function
4.3.2	Properties of the sine function
4.3.3	The inverse sine function and its properties
4.4	The Cosine Function and Inverse Cosine Function
4.4.2	Properties of the cosine function
4.4.3	The inverse cosine function and its properties
4.5	The Tangent Function and the Inverse Tangent Function
4.5.2	Properties of the tangent function
4.5.3	The inverse tangent function and its properties
4.6	The Cosecant Function and the Inverse Cosecant Function
4.6.2	The inverse cosecant function
4.7	The Secant Function and Inverse Secant Function
4.7.2	Inverse secant function
4.8	The Cotangent Function and the Inverse Cotangent Function
4.8.2	Inverse cotangent function
4.9	Principal Value of Inverse Trigonometric Functions (*All properties without proof)
5.	Two Dimensional Analytical Geometry-ll
5.1	Introduction (Theorem 5.1-5.5 without proof)
5.2	Circle
5.2.1	Equation of a circle in standard form
5.2.2	Equations of tangent and normal at a point P on a given circle (without proof)
5.2.3	Condition for the line y = $mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ and finding the point of contact (without proof)
5.3	Conics
5.3.1	The general equation of a Conic
5.3.2	Parabola
5.3.3	Ellipse (Theorem 5.3.3-without proof)
5.3.4	Hyperbola (Theorem 5.3.4-without proof)
5.4	Conic Sections
5.4.1	Geometric description of conic section
5.4.2	Degenerate Forms
5.5	Parametric form of Conics

UNIT	CONTENT
5.5.1	Parametric equations
5.6	Tangents and Normals to Conics
5.6.1	Equation of tangent and normal to the parabola $y^2 = 4ax$ (without proof)
5.6.2	Equations of tangent and normal to Ellipse and Hyperbola (without proof)
5.6.3	Condition for the line y = mx + c to be a tangent to the conic sections (without proof)
5.7	Real life Applications of Conics
5.7.1	Parabola
5.7.2	Ellipse
5.7.3	Hyperbola
5.7.4	Reflective property of parabola
5.7.5	Reflective property of Ellipse (*All properties without proof)
6.	Applications of Vector Algebra
6.1	Introduction (Theorems 6.1 -6.23-without proof)
6.2	Geometric Introduction to Vectors
6.3	Scalar Product and Vector Product
6.3.1	Geometrical interpretation
6.3.2	Application of dot and cross products in plane Trigonometry
6.3.3	Application of dot and cross products in Geometry
6.3.4	Application of dot and cross product in Physics
6.4	Scalar triple product
6.4.1	Properties of the scalar triple product
6.5	Vector triple product
6.6	Jacobi's Identity and Lagrange's Identity
6.7	Application of Vectors to 3D Geometry
6.7.1	Different forms of equation of a straight line
6.7.2	A point on the straight line and the direction of the straight line are given
6.7.3	Straight Line passing through two given points
6.7.4	Angle between two straight lines
6.7.5	Point of intersection of two straight lines
6.7.6	Shortest distance between two straight lines
6.8	Different forms of Equation of a plane
6.8.1	Equation of a plane when a normal to the plane and the distance of the plane from the origin are given

Strictly as per the Reduced (Prioritised) Syllabus released on 13th August, 2021 (G.O.(Ms).No126)

UNIT	CONTENT
6.8.2	Equation of a plane perpendicular to a vector and passing through a given point
6.8.3	Intercept form of the equation of a plane
6.8.4	Equation of a plane passing through three given non-collinear points
6.8.5	Equation of a plane passing through a given point and parallel to two given non-parallel vectors
6.8.6	Equation of a plane passing through two given distinct points and is parallel to a non-zero vector
6.8.7	Condition for a line to lie in a plane
6.8.8	Condition for coplanarity of two lines
6.8.10	Angle between two planes
6.8.11	Angle between a line and a plane
6.8.12	Distance of a point from a plane
6.8.13	Distance between two parallel planes (*All properties without proof)
7.	Applications of Differential Calculus
7.1	Introduction
7.1.1	Early Developments
7.2	Meaning of Derivatives
7.2.1	Derivative as slope
7.2.2	Derivative as rate of change
7.2.3	Related rates
7.2.4	Equations of Tangent and Normal
7.2.5	Angle between two curves
7.5	Indeterminate Forms
7.5.1	A Limit Process
7.5.2	The I'Hopital's Rule
7.5.3	Indeterminate forms $\left(\frac{0}{0},\frac{\infty}{\infty},0\times\infty,\infty-\infty\right)$
7.6	Applications of First Derivative
7.6.1	Monotonicity of functions
7.6.2	Absolute maxima and minima
7.6.3	Relative Extrema on an Interva
7.6.4	Extrema using First Derivative Test
7.7	Applications of Second Derivative
7.7.1	Concavity, Convexity, and Points of Inflection
7.7.2	Extrema using Second Derivative Test
7.8	Applications in Optimization (*All properties without proof)

UNIT	CONTENT
8.	Differentials and Partial Derivatives
8.1	Introduction
8.2	Linear Approximation and Differentials
8.2.2	Errors: Absolute Error, Relative Error, and Percentage Error
8.2.3	Differentials (*All properties without proof)
9.	Applications of integration
9.1	Introduction
9.3	Fundamental Theorems of Integral Calculus and their Applications
9.5	Improper Integrals
9.6	Reduction Formulae
9.7	Gamma Integral
9.8	Evaluation of Bounded Plane Area by Integration
9.8.1	Area of the region bounded by a curve, x - axis and the lines $x = a$ and $x = b$.
9.8.2	Area of the region bounded by a curve, y- axis and the lines $y = c$ and $y = d$.
9.8.3	Area of the region bounded between two curves (*All properties without proof)
10.	Ordinary Differential Equations
10.1	Introduction
10.2	Differential Equation, Order, and Degree
10.4	Formation of Differential Equations
10.4.1	Formation of Differential equations from Physical Situations
10.4.2	Formation of Differential Equations from Geometrical Problems
10.5	Solution of Ordinary Differential Equations
10.6	Solution of First Order and First Degree Differential Equations
10.6.1	Variables Separable Method
10.6.3	Homogeneous Form or Homogeneous Differential Equation
10.7	First Order Linear Differential Equations
10.8	Applications of First Order Ordinary Differential Equations
10.8.1	Population growth
10.8.2.	Radioactive decay

Strictly as per the Reduced (Prioritised) Syllabus released on 13th August, 2021 (G.O.(Ms).No126)

UNIT	CONTENT
10.8.3.	Newton's Law of cooling/warming
10.8.4	Mixture problems
11.	Probability Distributions
11.1	Introduction
11.2	Random Variable
11.3	Types of Random Variable
11.3.1	Discrete random variables
11.3.2	Probability Mass Function
11.3.3	Cumulative Distribution Function or Distribution Function
11.3.4	Cumulative Distribution Function from Probability Mass function
11.3.5	Probability Mass Function from Cumulative Distribution Function
11.4	Continuous Distributions
11.4.1	The definition of continuous random variable
11.4.2	Probability density function
11.4.3	Distribution function (Cumulative distribution function)
11.4.4	Distribution function from Probability density function

UNIT	CONTENT
11.4.5	Probability density function from Probability distribution function (*All properties without proof)
12.	Discrete Mathematics
12.1	Introduction
12.2	Binary Operations
12.2.1	Definitions
12.2.2	Some more properties of a binary operation
12.2.3	Some binary operations on Boolean Matrices
12.2.4	Modular Arithmetic
12.3	Mathematical Logic
12.3.1	Statement and its truth value
12.3.2	Compound Statements, Logical Connectives, and Truth Tables
12.3.3	Tautology, Contradiction, and Contingency
12.3.4	Duality
12.3.5	Logical Equivalence (*All properties without proof) (*All examples and exercise problems for the content mentioned above)

CONTENTS

Volume - I

Volume - II

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CONTENTS

VOLUME - I

CHAPTER 1

APPLICATIONS OF MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

- If $|A| \neq 0$, then A is a non-singular matrix and if $|A| = 0$, then A is a singular matrix.
- The adjoint matrix of A is defined as the transpose of the matrix of co-factors of A.
- If $AB = BA = I_n$, then the matrix B is called the inverse of A.
- If a square matrix has an inverse, then it is unique.
- A^{-1} exists if and only if A is non-singular.
- Singular matrix has no inverse.
- If A is non – singular and AB = AC, then B = C (left cancellation law).
- If A is non – singular and BA = CA then B = C (Right cancellation law).
- If A and B are any two non-singular square matrices of order n, then adj (AB) = (adj B) (adj A)
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- Two matrices A and B of same order are said to the **equivalent** if one can be obtained from the other by the applications of elementary transformations (A~B).
- A non – zero matrix is in a **row - echelon f**orm if all zero rows occur as bottom rows of the matrix and if the first non – zero element in any lower row occurs to the right of the first non – zero entry in the higher row.
- The **rank** of a matrix A is defined as the order of a highest order non – vanishing minor of the matrix A [$\rho(A)$].
- The **rank** of a non – zero matrix is equal to the number of non – zero rows in a row – echelon form of the matrix.
- An **elementary matrix** is a matrix which is obtained from an identity matrix by applying only one elementary transformation. Every non-singular matrix can be transformed to an identity matrix by a sequence of elementary row operations..

IMPORTANT FORMULAE TO REMEMBER

- Co – factor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}
- For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$
  $AA^{-1} = A^{-1}A = I_n$
- If A is non – Singular then
  (i) $|A^{-1}| = \dfrac{1}{|A|}$
  (ii) $(A^T)^{-1} = (A^{-1})^T$
  (iii) $(\lambda A^{-1}) = \dfrac{1}{\lambda} A^{-1}$ where λ is a non – zero scalar.

Reversal law for inverses :

- $(AB)^{-1} = B^{-1}A^{-1}$ where A, B are non – singular matrices of same order.

Law of double inverse :

- If A is non - singular, A^{-1} is also non – singular and $(A^{-1})^{-1} = A$.
- If A is a non - singular square matrix of order n, then
  (i) $(\text{adj } A)^{-1} = \text{adj } (A^{-1}) = \dfrac{1}{|A|} \cdot A$
  (ii) $|\text{adj } A| = |A|^{n-1}$
  (iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$
  (iv) $\text{adj } (\lambda A) = \lambda^{n-1} \text{adj } (A)$ where λ is a non – zero scalar
  (v) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$
  (vi) $(\text{adj } A)^T = \text{adj } (A^T)$
- If a matrix contains at least one non – zero element, then $\rho(a) \geq 1$.
- The rank of identity matrix I_n is n.
  If A is an $m \times n$ matrix then $\rho(A) \leq \min\{m, n\}$.
- A square matrix A of order n is invertible if and only if $\rho(A) = n$.
- Transforming a non-singular matrix A to the form I_n, by applying row operations is called Gauss – Jordan method.

Matrix – Inversion method :

- The solution for $AX = B$ is $X = A^{-1}B$ where A and B are square matrices of same order and non – singular

Cramer's Rule :

- If $\Delta = 0$, Cramer's rule cannot be applied $x_1 = \dfrac{\Delta_1}{\Delta}$, $x_2 = \dfrac{\Delta_2}{\Delta}$, $x_3 = \dfrac{\Delta_3}{\Delta}$

EXERCISE 1.1

1. Find the adjoint of the following :

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ **(iii)** $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Sol. **(i)** Let $A = \begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

$$\text{adj } A = \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$$

[Interchange the elements in the leading diagonal and change the sign of the elements in off diagonal]

(ii) Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$$\text{adj } A = \begin{pmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $\lambda = \frac{1}{3}$

Since $\text{adj }(\lambda A) = \lambda^{n-1}(\text{adj } A)$

we get adj $\left(\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\right) = \left(\frac{1}{3}\right)^2$

adj $\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

$\therefore$ Required adjoint matrix

$$= \frac{1}{9}\begin{bmatrix} +\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} (2+4) & -(-4-2) & +(4-1) \\ -(4+2) & +(4-1) & -(-4-2) \\ +(4-1) & -(4+2) & +(2+4) \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{3}{9}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

[Taking 3 common from each entry]

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse (if it exists) of the following :

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Sol. **(i)** Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Since A is non – singular, A^{-1} exists

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

Now, $\text{adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

[Inter change the entries in leading diagonal and change the sign of elements in the off diagonal]

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

(ii) Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Expanding along R_1,

$$|A| = 5\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$
$$= 5(25-1) - 1(5-1) + 1(1-5)$$
$$= 5(24) - 1(4) + 1(-4)$$
$$= 120 - 4 - 4 = 120 - 8 = 112 \neq 0$$

Since A is non singular, A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(25-1)-(5-1)+(1-5) \\ -(5-1)+(25-1)-(5-1) \\ +(1-5)-(5-1)+(25-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

Taking 4 common from every entry we get,

$$\text{adj } A = 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{112} \cdot 4 \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$= \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Expanding along R_1 we get,

$$|A| = 2\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$
$$= 2(8-7) - 3(6-3) + 1(21-12)$$
$$= 2(1) - 3(3) + 1(9)$$
$$= 2 - \not{9} + \not{9} = 2 \neq 0$$

Since A is a non-singular matrix, A^{-1} exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(8-7)-(6-3)+(21-12) \\ -(6-7)+(4-3)-(14-9) \\ +(3-4)-(2-3)+(8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. **If** $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, **show that** $[F(\alpha)]^{-1} = F(-\alpha)$ **[Hy - 2019]**

Sol. Given that $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$.

Expanding along R_1 we get,

$$|F(\alpha)| = \cos\alpha \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} - 0 + \sin\alpha \begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix}$$

$= \cos\alpha(\cos - 0) + \sin\alpha(0 + \sin\alpha)$

$= \cos^2 + \sin^2\alpha = 1 \neq 0$

Since $F(\alpha)$ is a non-singular matrix, $[F(\alpha)]^{-1}$ exists.

Now, adj (F(α)) =

$$\begin{bmatrix} +\begin{vmatrix}1 & 0\\0 & \cos\alpha\end{vmatrix} & -\begin{vmatrix}0 & 0\\\sin\alpha & \cos\alpha\end{vmatrix} & +\begin{vmatrix}0 & 1\\-\sin\alpha & 0\end{vmatrix} \\ -\begin{vmatrix}0 & \sin\alpha\\0 & \cos\alpha\end{vmatrix} & +\begin{vmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{vmatrix} & -\begin{vmatrix}\cos\alpha & 0\\-\sin\alpha & 0\end{vmatrix} \\ +\begin{vmatrix}0 & \sin\alpha\\1 & 0\end{vmatrix} & -\begin{vmatrix}\cos\alpha & \sin\alpha\\0 & 0\end{vmatrix} & +\begin{vmatrix}\cos\alpha & 0\\0 & 1\end{vmatrix}\end{bmatrix}^T$$

$$= \begin{bmatrix} +(\cos\alpha - 0) & -(0) & +(0+\sin\alpha) \\ -(0) & +(\cos^2\alpha + \sin^2\alpha) & -(0) \\ +(0-\sin\alpha) & -(0) & +(\cos - 0)\end{bmatrix}^T$$

$$= \begin{bmatrix}\cos\alpha & 0 & +\sin\alpha\\0 & 1 & 0\\-\sin\alpha & 0 & \cos\alpha\end{bmatrix}^T = \begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\0 & 1 & 0\\\sin\alpha & 0 & \cos\alpha\end{bmatrix}$$

$$\therefore F(\alpha)^{-1} = \frac{1}{|F(\alpha)|}\text{adj } (F(\alpha))$$

$$[F(\alpha)]^{-1} = \frac{1}{1}\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\0 & 1 & 0\\\sin\alpha & 0 & \cos\alpha\end{bmatrix}$$

$$= \begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\0 & 1 & 0\\\sin\alpha & 0 & \cos\alpha\end{bmatrix} \quad ...(1)$$

$$\text{Now, } F(-\alpha) = \begin{bmatrix}\cos(-\alpha) & 0 & \sin(-\alpha)\\0 & 1 & 0\\-\sin(-\alpha) & 0 & \cos(-\alpha)\end{bmatrix}$$

$$= \begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\0 & 1 & 0\\\sin\alpha & 0 & \cos\alpha\end{bmatrix} \quad ...(2)$$

[∵ cosα is an even function, cos (–α) = cos α and sin α is an odd function, sin (–α) = –sinα]

From (1) and (2)

$$[F(\alpha)]^{-1} = F(-\alpha)$$

Hence proved.

4. If $A=\begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}.

Sol. Given $A = \begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix}$

$$A^2 = \begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix}\begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix} = \begin{bmatrix}25-3 & 15-6\\-5+2 & -3+4\end{bmatrix}$$

$$= \begin{bmatrix}22 & 9\\-3 & 1\end{bmatrix}$$

$\therefore A^2 - 3A - 7I_2$

$$= \begin{bmatrix}22 & 9\\-3 & 1\end{bmatrix} - 3\begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix} - 7\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}22-15-7 & 9-9+0\\-3+3+0 & 1+6-7\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0_2$$

Hence proved.

$\therefore A^2 - 3A - 7I_2 = 0$

Post – multiplying by A^{-1} we get,

$A^2.A^{-1} - 3AA^{-1} - 7I_2A^{-1} = 0.A^{-1}$

$\Rightarrow A(AA^{-1}) - 3(AA^{-1}) - 7(A^{-1}) = 0$

[∵ $I_2A^{-1} = A^{-1}$ and $(0)A^{-1} = 0$]

$\Rightarrow AI - 3I - 7A^{-1} = 0$ [∵ $AA^{-1} = I$]

$\Rightarrow AI - 3I = 7A^{-1}$

$\Rightarrow A^{-1} = \frac{1}{7}[A - 3I]$ [∴ AI = A]

$$\Rightarrow A^{-1} = \frac{1}{7}\left[\begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix} - 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix}5-3 & 3-0\\-1-0 & -2-3\end{bmatrix} = \frac{1}{7}\begin{bmatrix}2 & 3\\-1 & -5\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix}2 & 3\\-1 & -5\end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\4 & 4 & 7\\1 & -8 & 4\end{bmatrix}$, prove that $A^{-1} = A^T$.

Sol. To prove $A^{-1} = A^T$

$AA^{-1} = AA^T$

It is enough to prove $AA^T = I$

$$AA^T = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\4 & 4 & 7\\1 & -8 & 4\end{bmatrix}\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\1 & 4 & -8\\4 & 7 & 4\end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32-28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \frac{81}{81}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore $A^{-1} = A^T$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A| I_2$. **[Sep. - 2020]**

Sol. Given $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$\text{adj } A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

[Interchange the elements in the leading diagonal and change the sign of the elements in the off diagonal]

$|A| = 24 - 20 = 4$

$\therefore A(\text{adj } A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{...(1)}$$

$(\text{adj } A)(A) = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{...(2)}$$

$$|A| I_2 = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{...(3)}$$

From (1), (2) and (3), it is proved that

$A(\text{adj } A) = (\text{adj } A)A = |A| I_2$ is verified.

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. Given $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$$

$|AB| = -77 + 90 = 13 \neq 0 \Rightarrow (AB)^{-1}$ exists

$|A| = 15 - 14 = 1 \neq 0 \Rightarrow A^{-1}$ exists

$|B| = -2 + 15 = 13 \neq 0 \Rightarrow B^{-1}$ exists

$$(AB)^{-1} = \frac{1}{|AB|}\text{adj}(AB) = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix} \quad \text{...(1)}$$

$$B^{-1} = \frac{1}{|B|}(\text{adj } B) = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$\therefore B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix} \quad \text{...(2)}$$

From (1) and (2) it is prove that

$(AB)^{-1} = B^{-1}A^{-1}$.

8. If $\text{adj}(A) = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

Sol. Given $\text{adj } A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

We know that $A = \pm\frac{1}{\sqrt{|\text{adj} A|}}.\text{adj}(\text{adj } A) \quad \text{...(1)}$

$$|\text{adj } A| = 2\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix}$$

[Expanded along R_1]

$= 2(24-0) + 4(-6-14) + 2(0+24)$

$= 2(24) + 4(-20) + 2(24) = 48 - 80 + 48$

$= 96 - 80 = 16$

Now, adj (adj A)

$$= \begin{bmatrix} +\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -4 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -2 & 0 \end{vmatrix} \\ +\begin{vmatrix} -4 & 2 \\ 12 & -7 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -3 & -7 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ -3 & 12 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(24-0)-(-6-14)+(0+24) \\ -(-8-0)+(4+4)-(0-8) \\ +(28-24)-(-14+6)+(24-12) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}^T = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$= 4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix} \quad ...(3)$$

Substituting (2) and (3) in (1) we get,

$$A = \pm\frac{1}{\sqrt{16}}.4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$A = \pm\frac{4}{4}\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix} = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

9. If adj(A) = $\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$, find A^{-1}. [PTA-6]

Sol. Given adj (A) = $\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

We know that $A^{-1} = \pm\frac{1}{\sqrt{|\text{adj}A|}}(\text{adj } A)$ ...(1)

$$|\text{adj } A| = 0 + 2\begin{vmatrix} 6 & -6 \\ -3 & 6 \end{vmatrix} + 0$$

[Expanded along R_1]

$$= 2\,(36 - 18) = 2(18) = 36$$

$$\therefore A^{-1} = \pm\frac{1}{\sqrt{36}}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix} = \pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

10. Find adj (adj (A)) if adj A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

Sol. Given adj A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now adj(adj A) = $\begin{bmatrix} +\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$

$$= \begin{bmatrix} +(2-0) & -(0) & +(0+2) \\ -(0) & +(1+1) & -(0) \\ +(0-2) & -(0) & +(2-0) \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T$$

$$\text{adj (adj } A) = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

11. A = $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + \tan^2 x = \sec^2 x$$

$$\text{adj } A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

We know, $A^{-1} = \frac{1}{|A|}\text{Adj } A$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} = \cos^2 x\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \cos^2 x\begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos^2 x & -\cos x\sin x \\ \cos x\sin x & \cos^2 x \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} 1 & -\dfrac{\sin x}{\cos x} \\ \dfrac{\sin x}{\cos x} & 1 \end{bmatrix}\begin{bmatrix} \cos^2 x & -\cos x \sin x \\ \cos x \sin x & \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -2\sin x \cos x \\ 2\sin x \cos x & \cos^2 x - \sin^2 x \end{bmatrix} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

12. Find the matrix A for which

$$\mathbf{A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}}.$$

Sol. Given $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Let $B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$\therefore AB = C$

Post multiply by B^{-1} we get

$A(BB^{-1}) = CB^{-1}$

$\Rightarrow A = CB^{-1}$ $[\because BB^{-1} = I]$

$$|B| = \begin{vmatrix} 5 & 3 \\ -1 & -2 \end{vmatrix} = -10 + 3 = -7 \neq 0$$

$\therefore B^{-1}$ exists

$$B^{-1} = \frac{1}{|B|} \text{ adj } B = \frac{-1}{7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$A = CB^{-1}$

$$= \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \cdot \left(\frac{-1}{7}\right)\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= 7\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\left(\frac{-1}{7}\right)\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= -\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= -\begin{bmatrix} -4+1 & -6+5 \\ -2+1 & -3+5 \end{bmatrix} = -\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that AXB = C.

Sol. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Also, $AXB = C$

Pre-multiply by A^{-1} we get,

$(A^{-1}A)XB = A^{-1}C$

$\Rightarrow XB = A^{-1}.C.$ $[\because A^{-1}A = I]$

Post Multiply by B^{-1} we get

$(XB)B^{-1} = (A^{-1}C)B^{-1}$

$\Rightarrow X = (A^{-1}C)B^{-1}$

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 = 2 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 + 2 = 5 \neq 0$$

$$\therefore B^{-1} = \frac{1}{|B|} \text{ adj } B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1}C = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore X = (A^{-1}C).B^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1-1 & 2+3 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \frac{1}{5}(5)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$.

Sol. Given $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = 0 - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$= -1(0-1) + 1(1-0) = 1 + 1 = 2$

$|A| = 2 \neq 0$, hence A^{-1} exists

$$\text{adj A} = \begin{bmatrix} +\begin{vmatrix}0&1\\1&0\end{vmatrix} & -\begin{vmatrix}1&1\\1&0\end{vmatrix} & +\begin{vmatrix}1&0\\1&1\end{vmatrix} \\ -\begin{vmatrix}1&1\\1&0\end{vmatrix} & +\begin{vmatrix}0&1\\1&0\end{vmatrix} & -\begin{vmatrix}0&1\\1&1\end{vmatrix} \\ +\begin{vmatrix}1&1\\0&1\end{vmatrix} & -\begin{vmatrix}0&1\\1&1\end{vmatrix} & +\begin{vmatrix}0&1\\1&0\end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (0-1) & -(0-1) & +(1-0) \\ -(0-1) & +(0-1) & -(0-1) \\ +(1-0) & -(0-1) & +(0-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad ...(1)$$

$$\text{Now } A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-3 & 1-0 & 1-0 \\ 1-0 & 2-3 & 1-0 \\ 1-0 & 1-0 & 2-3 \end{bmatrix}$$

$$= \frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad ... (2)$$

From (1) and (2), it is proved that $A^{-1} = \frac{1}{2}[A^2 - 3I]$

EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$ **[PTA - 5]**

(iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$ **(v)** $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Sol. **(i)** Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

A is a matrix of order 2 × 2

$\therefore \rho(A) \leq \min(2, 2) = 2$

The highest order of minor of A is 2

It is $\begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 4 - 4 = 0$

So, $\rho(A) < 2$

Next consider the minor of order 1 $|2| = 2 \neq 0$

$\therefore \rho(A) = 1$

(ii) Let $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

A is a matrix of order 3 × 2

$\therefore \rho(A) \leq \; = 2$

We find that there is a second order minor,

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$

$\therefore \rho(A) = 2.$

(iii) Let $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

A is a matrix of order (2 × 4)

$\therefore \rho(A) \leq \min(2, 4) = 2$

The highest order of minor of A is 2

It is $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6 + 6 = 0$

Also, $\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1 + 0 = -1 \neq 0.$

$\therefore \rho(A) = 2.$

(iv) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

A is a matrix of order 3 × 3

$\therefore \rho(A) \leq \min(3, 3) = 3$

The highest order of minor of A is 3.

It is $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$

[Expanded along R_1]

$= 1(-4 + 6) + 2(-2 + 30) + 3(2 - 20)$

$= 1(2) + 2(28) + 3(-18)$

$= 2 + 56 - 54 = 58 - 54 = 4 \neq 0$

$\therefore \rho(A) = 3.$

(v) Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

A is a matrix of order 3 × 4

$\therefore \rho(A) \leq \min(3, 4) = 3$

The highest order of minor of A is 3

It is $\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 8 & 1 & 0 \end{vmatrix} = 0 + 0 + 8(4 - 4) = 0$

[Expanded along C_1]

Also, $\begin{vmatrix} 0 & 2 & 1 \\ 0 & 4 & 3 \\ 8 & 0 & 2 \end{vmatrix} = 0 + 0 - 8 \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$

[Expanded along C_1]

$= -8(6 - 4) = -8(2) = -16 \neq 0$

$\therefore \rho(A) = 3$

2. Find the rank of the following matrices by row reduction method :

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ **[PTA-1]**

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Sol. **(i)** Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 5 & -1 & 7 & 11 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 5R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in row echelon form it has two non-zero rows

$\therefore \rho(A) = 2.$

(ii) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 \div 4} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 3R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 7R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_4 \to 2R_4 + R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in row echelon form it has two non-zero rows

$\therefore \rho(A) = 3$

(iii) Let $A = \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

$$A \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 + 2R_1} \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 \div 2} \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -1 & 7 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The last equivalent matrix is in row-echelon form. It has three non-zero rows.

$\therefore \rho(A) = 3$

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

(i) $2x + 5y = -2,\ x + 2y = -3$

(ii) $2x - y = 8,\ 3x + 2y = -2$ **[PTA -3]**

(iii) $2x + 3y - z = 9,\ x + y + z = 9,\ 3x - y - z = -1$

(iv) $x + y + z - 2 = 0,\ 6x - 4y + 5z - 31 = 0,\ 5x + 2y + 2z = 13.$

Sol. **(i)** $2x + 5y = -2,\ x + 2y = -3$

The matrix form of the system is

$$= \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$\Rightarrow$ AX = B where

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix},\ B = \begin{pmatrix} -2 \\ -3 \end{pmatrix},$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ $X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1 \neq 0.$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-15 \\ -2+6 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$\therefore x = -11,\ y = 4.$

(ii) $2x - y = 8,\ 3x + 2y = -2$

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ AX = B where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$,

$$B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ $X = A^{-1}B.$

$$\text{Now, } |A| = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = 4 + 3 = 7$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} \frac{14}{7} \\ \frac{-28}{7} \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$\therefore x = 2,\ y = -4$

(iii) $2x + 3y - z = 9,\ x + y + z = 9,\ 3x - y - z = -1.$

The matrix form of the system is

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$\Rightarrow$ AX = B where $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$\Rightarrow$ $X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$$

[Expanded along R_1]

$= 2(-1+1) - 3(-1-3) - 1(-1-3)$

$= 0 - 3(-4) - 1(-4) = 12 + 4 = 16.$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-1+1) & -(-1-3) & +(-1-3) \\ -(-3-1) & +(-2+3) & -(-2-9) \\ +(3+1) & -(2+1) & +(2-3) \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$\therefore X = A^{-1} B = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\therefore x = 2, y = 3, z = 4$

(iv) $\boldsymbol{x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13}$

The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$AX = B \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1} B$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix}$$

$= 1(-8-10) - 1(12-25) + 1(12+20)$

$= 1(-18) - 1(-13) + 1(32) = -18 + 13 + 32 = 27$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} & +\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-8-10) & -(12-25) & +(12+20) \\ -(2-2) & +(2-5) & -(2-5) \\ +(5+4) & -(5-6) & +(-4-6) \end{bmatrix}^T$$

$$= \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}^T = \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\therefore X = A^{-1} B = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -36+0+117 \\ 26-93+13 \\ 64+93-130 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore x = 3, y = -2, z = 1$

2. **If** $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, **find the products AB and BA and hence solve the system of equations** $\boldsymbol{x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2.}$

Sol. Given $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4.\, I_3$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4.\ I_3.$$

So, we get $AB = BA = 4.\ I_3$

$$\Rightarrow \left(\frac{1}{4}A\right)B = B\left(\frac{1}{4}A\right) = I$$

$$\Rightarrow B^{-1} = \frac{1}{4}A$$

Writing the given set of equations in matrix form we get,

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\Rightarrow B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \left[\frac{1}{4}A\right]\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore x = 2, y = 1, z = -1$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹ 19,800 per month at the end of the first month after 3 years of service and ₹ 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

Sol. Let the man's starting salary be ₹ x and his annual increment be ₹ y.

By the given data $x + 3y = 19800$ and $x + 9y = 23{,}400$.

The matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$\Rightarrow AX = B \text{ where } A = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = 9 - 3 = 6 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 178200 - 70200 \\ -19800 + 23400 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 108000 \\ 3600 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$$

$\therefore x = 18000, y = 600.$

Hence the man's starting salary is ₹ 18000 and his annual increment is ₹ 600.

4. Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Sol. Let the time by one man alone be x days and one woman alone be y days

$\therefore$ By the given data,

$$\frac{4}{x} + \frac{4}{y} = \frac{1}{3} \text{ and } \frac{2}{x} + \frac{5}{y} = \frac{1}{4}$$

$$\text{put } \frac{1}{x} = s \text{ and } \frac{1}{y} = t$$

$$\therefore 4s + 4t = \frac{1}{3} \text{ and } 2s + 5t = \frac{1}{4}$$

The matrix form of the system of equation is

$$\begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \Rightarrow AX = B \text{ where}$$

$$A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$X = A^{-1}B$$

Now $|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix} = 20 - 8 = 12 \neq 0$

$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

$\therefore X = A^{-1}B = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{5}{3}-1 \\ \frac{-2}{3}+1 \end{bmatrix}$

$= \frac{1}{12}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\times\frac{1}{12} \\ \frac{1}{3}\times\frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$

$\therefore s = \frac{1}{18} \Rightarrow \frac{1}{x} = \frac{1}{18} \Rightarrow x = 18$

$t = \frac{1}{36} \Rightarrow \frac{1}{y} = \frac{1}{36} \Rightarrow y = 36.$

Hence, the time taken by 1 man alone is 18 days and the time taken by 1 woman alone is 36 days.

5. The prices of three commodities A, B and C are ₹ x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P,Q and R earn ₹ 15,000, ₹ 1,000 and ₹ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Sol. Let the prices per unit for the commodities A, B and C be ₹ x, ₹ y and ₹ z.

By the given data,

$$2x - 4y + 5z = 15000$$
$$3x + y - 2z = 1000$$
$$-x + 3y + z = 4000$$

The matrix form of the system of equations is

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

AX = B where $A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

and $B = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$

$\Rightarrow X = A^{-1}B$

$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}$

$= 2(1 + 6) + 4(3 - 2) + 5(9 + 1)$

$= 2(7) + 4(1) + 5(10) = 14 + 4 + 50 = 68.$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -4 & 5 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -1 & 3 \end{vmatrix} \\ +\begin{vmatrix} -4 & 5 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(1+6) & -(3-2) & +(9+1) \\ -(-4-15) & +(2+5) & -(6-4) \\ +(8-5) & -(-4-15) & +(2+12) \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 105000+19000+12000 \\ -15000+7000+76000 \\ 150000-2000+56000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$\therefore x = 2000, y = 1000, z = 3000.$

Hence the prices per unit of the commodities A, B and C are ₹ 2000, ₹ 1000 and ₹ 3000 respectively.

EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

(i) $5x - 2y + 16 = 0,\ x + 3y - 7 = 0$

(ii) $\frac{3}{x} + 2y = 12,\ \frac{2}{x} + 3y = 13$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9,$
$4x + 3y + 2z = 25$ [Hy - 2019]

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,$
$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Sol. (i) Given $\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$

$$\Delta_1 = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_2 = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-34}{17} = -2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{51}{17} = 3$$

$x = -2, y = 3.$

(ii) Let $\frac{1}{x} = z$

$$\therefore 3z + 2y = 12,\ 2z + 3y = 13$$

$$\therefore \Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$$

$$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

$$\therefore z = \frac{\Delta_1}{\Delta} = \frac{10}{5} = 2 \Rightarrow \frac{1}{x} = 2 \Rightarrow x = \frac{1}{2}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$$

$$\therefore x = \frac{1}{2}, y = 3$$

(iii) $$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3(-2-6) - 3(4-8) - 1(6+4)$$
$$= 3(-8) - 3(-4) - 1(10)$$
$$= -24 + 12 - 10 = -22$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$$= 11\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 1\begin{vmatrix} 9 & -1 \\ 25 & 3 \end{vmatrix}$$

$$= 11(-2-6) - 3(18-50) - 1(27+25)$$
$$= 11(-8) - 3(-32) - 1(52)$$
$$= -88 + 96 - 52 = -44$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 11\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix}$$

$$= 3(18-50) - 11(4-8) - 1(50-36)$$
$$= 3(-32) - 11(-4) - 1(14)$$
$$= -96 + 44 - 14 = -66$$

$$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 9 \\ 3 & 25 \end{vmatrix} - 3\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix} + 11\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3(-25-27) - 3(50-36) + 11(6+4)$$
$$= 3(-52) - 3(14) + 11(10)$$
$$= -156 - 42 + 110 = -88$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-44}{-22} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-66}{-22} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-88}{-22} = 4$$

$\therefore x = 2, y = 3, z = 4.$

(iv) Put $\frac{1}{x} = X, \frac{1}{y} = Y, \frac{1}{z} = Z$

We get $3X - 4Y - 2Z = 1$, $X + 2Y + Z = 2$,
$2X - 5Y - 4Z = -1$

$$\therefore \Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$= 3(-8+5) + 4(-4-2) - 2(-5-4)$
$= 3(-3) + 4(-6) - 2(-9)$
$= -9 - 24 + 18 = -15$

$$\Delta_x = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & -5 \end{vmatrix}$$

$= 1(-8+5) + 4(-8+1) - 2(-10+2)$
$= 1(-3) + 4(-7) - 2(-8)$
$= -3 - 28 + 16 = -15$

$$\Delta_y = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$= 3(-8+1) - 1(-4-2) - 2(-1-4)$
$= 3(-7) - 1(-6) - 2(-5)$
$= -21 + 6 + 10 = -5$

$$\Delta_z = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$= 3(-2+10) + 4(-1-4) + 1(-5-4)$
$= 3(8) + 4(-5) + 1(-9)$
$= 24 - 20 - 9 = -5$

$$\therefore X = \frac{\Delta_x}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

$$Y = \frac{\Delta_y}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3$$

$$Z = \frac{\Delta_z}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{z} = \frac{1}{3} \Rightarrow z = 3$$

$\therefore x = 1, y = 3, z = 3.$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem). **[Qy - 2019]**

Sol. Let x represent the number of question with correct answer and y represent the number of questions with wrong answers.

By the given data, $x + y = 100$ and ... (1)

$$1.x - \frac{1}{4}y = 80$$

Multiplying by 4 we get,

$$4x - y = 320 \quad \text{... (2)}$$

From (1) and (2)

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1 - 4 = -5$$

$$\Delta_1 = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100 - 320 = -420$$

$$\Delta_2 = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix} = 320 - 400 = -80$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-420}{-5} = +84$$

$$\text{and } y = \frac{\Delta_2}{\Delta} = \frac{-80}{-5} = 16$$

Hence, the number of questions with correct answer is 84.

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

Sol. Let the amount of 50% acid be x litres and the amount of 25% acid be y litres

By the given data, $x + y = 10$ ... (1)

$$\text{and } x\left(\frac{50}{100}\right) + y\left(\frac{25}{100}\right) = 10\left(\frac{40}{100}\right)$$

$$\Rightarrow \quad 50x + 25y = 400 \Rightarrow 2x + y = 16 \quad \text{... (2)}$$

$$\Delta = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = -1$$

$$\Delta_x = \begin{vmatrix} 10 & 1 \\ 16 & 1 \end{vmatrix} = 10 - 16 = -6$$

$$\Rightarrow \quad \Delta_y = \begin{vmatrix} 1 & 10 \\ 2 & 16 \end{vmatrix} = 16 - 20 - 4$$

$$x = \frac{\Delta_x}{\Delta_y} = \frac{-6}{-1} = 6$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-4}{-1} = 4$$

i.e., 6 litres of 50% acid and 4 litres of 25% acid solution to be mixed to get 10 litres of 40% of acid solution.

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

Sol. Let the pump A can fill the tank in x minutes, and the pump B can fill the tank in y minutes

In 1 minute A can fill $\frac{1}{x}$ units and in 1 minute B can fill $\frac{1}{y}$ units

$$\therefore \frac{1}{x} + \frac{1}{y} = 10$$

$$\text{and } \frac{1}{x} - \frac{1}{y} = 30$$

$$\text{Put } \frac{1}{x} = a \text{ and } \frac{1}{y} = b$$

$$\Rightarrow \quad a + b = \frac{1}{10} \quad \ldots (1)$$

$$\text{and } a - b = \frac{1}{30} \quad \ldots (2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$\Delta_1 = \begin{vmatrix} \frac{1}{10} & 1 \\ \frac{1}{30} & -1 \end{vmatrix} = \frac{-1}{10} - \frac{1}{30}$$

$$= \frac{-3-1}{30} = \frac{-4}{30} = \frac{-2}{15}$$

$$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{10} \\ 1 & \frac{1}{30} \end{vmatrix} = \frac{1}{30} - \frac{1}{10} = \frac{1-3}{30}$$

$$= \frac{-2}{30} = \frac{-1}{15}$$

$$\therefore a = \frac{\Delta_1}{\Delta} = \frac{\frac{-2}{15}}{-2} = \frac{1}{15} \Rightarrow \frac{1}{x} = \frac{1}{15} \Rightarrow x = 15$$

$$b = \frac{\Delta_2}{\Delta} = \frac{\frac{-1}{15}}{-2} = \frac{1}{30} \Rightarrow \frac{1}{y} = \frac{1}{30} \Rightarrow y = 30$$

Hence the pump A can fill the tank in 15 minutes and the pump B can fill the tank in 30 minutes.

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹ 150. The cost of the two dosai, two idlies and four vadais is ₹ 200. The cost of five dosai, four idlies and two vadais is ₹ 250. The family has ₹ 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

Sol. Let the cost of one dosa be ₹ x

The cost of one idli be ₹ y

and the cost of one vadai be ₹ z

By the given data,

$$2x + 3y + 2z = 150$$
$$2x + 2y + 4z = 200$$
$$5x + 4y + 2z = 250$$

$$\therefore \Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$= 2(4 - 16) - 3(4 - 20) + 2(8 - 10)$

$= 2(-12) - 3(-16) + 2(-2)$

$= -24 + 48 - 4 = 20$

$$\Delta_1 = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

Taking 50 common from C_1 and 2 common from C_3 we get,

$$= 100\begin{vmatrix} 3 & 3 & 1 \\ 4 & 2 & 2 \\ 5 & 4 & 1 \end{vmatrix} = 100\left[3\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix}\right]$$

$= 100[3(2-8) - 3(4-10) + 1(16-10)]$
$= 100[3(-6) - 3(-6) + 6]$
$= 100[-18 + 18 + 6] = 600.$

$$\Delta_2 = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix} = 100 \begin{vmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 5 & 5 & 1 \end{vmatrix}$$

$$= 100 \left[2\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix} \right]$$

$= 100[2(4-10) - 3(2-10) + 1(10-20)]$
$= 100[2(-6) - 3(-8) + 1(-10)]$
$= 100[-12 + 24 - 10] = 100\,[2] = 200.$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix} = 50 \begin{vmatrix} 2 & 3 & 3 \\ 2 & 2 & 4 \\ 5 & 4 & 5 \end{vmatrix}$$

$$= 50 \left[2\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix} \right]$$

$= 50\,[2(10-16) - 3(10-20) + 3(8-10)]$
$= 50[2(-6) - 3(-10) + 3(-2)]$
$= 50\,[-12 + 30 - 6] = 50\,[12] = 600.$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{600}{20} = 30$$

$$y = \frac{\Delta_2}{\Delta} = \frac{200}{20} = 10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{600}{20} = 30.$$

Hence, the price of one dosa be ₹ 30, one idli be ₹10 and the price of 1 vadai be ₹ 30.

Also the cost of 3 dosa, six idlies and six vadai is

$= 3x + 6y + 6z = 3(30) + 6(10) + 6(30)$
$= 90 + 60 + 180 =$ ₹ 330

Since the family had ₹ 350 in hand, they will be able to manage to pay the bill.

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method :

(i) $2x - 2y + 3z = 2, x + 2y - z = 3, 3x - y + 2z = 1.$

(ii) $2x + 4y + 6z = 22, 3x + 8y + 5z = 27,$
$-x + y + 2z = 2$

Sol. (i) Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -1 & 0 & -4 \end{array}\right] \xrightarrow{R_3 \to 6R_3 - 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right]$$

Writing the equivalent equations from the row-echelon matrix, we get,

$x + 2y - z = 3$ ... (1)
$-6y + 5z = -4$ ... (2)
$-5z = -20 \Rightarrow z = \frac{-20}{-5} = 4.$

Substituting $z = 4$ in 2 we get,

$-6y + 5(4) = -4$
$\Rightarrow -6y + 20 = -4 \Rightarrow -4-20 = -24$
$\Rightarrow y = \frac{-24}{-6} = 4.$

Substituting $y = z = 4$ in (1) we get

$x + 2(4) - 4 = 3$
$\Rightarrow x + 8 - 4 = 3$
$\Rightarrow x + 4 = 3$
$\Rightarrow x = 3 - 4 = -1.$

$\therefore x = -1, y = 4, z = 4.$

(ii) $2x + 4y + 6z = 22, 3x + 8y + 5z = 27,$
$-x + y + 2z = 2$

Reducing the augmented matrix to an equivalent row echelon form by using elementary row operations, we get

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & 8 & 5 & 27 \\ 2 & 4 & 6 & 22 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + 3R_1} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 11 & 11 & 33 \\ 0 & 6 & 10 & 26 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 2]{R_2 \to R_2 \div 11} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 \div 2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Writing the equivalent equations from the row echelon matrix we get,

$-x + y + 2z = 2$ ...(1)

$y + z = 3$ ...(2)

$z = 2$ ...(3)

Substituting (3) in (2) we get, $y + 2 = 3$

$\Rightarrow y = 3 - 2 = 1$

Substituting $y = 1$ and $z = 2$ in (1) we get,

$-x + 1 + 2(2) = 2 \quad \Rightarrow -x + 1 + 4 = 2$

$\Rightarrow -x + 5 = 2 \quad \Rightarrow -x = 2 - 5$

$\Rightarrow -x = -3 \quad \Rightarrow x = 3$

$\therefore x = 3, y = 1, z = 2.$

2. If $ax^2 + bx + c$ is divided by $x + 3$, $x - 5$, and $x - 1$, the remainders are 21, 61 and 9 respectively. Find a, b and c. (Use Gaussian elimination method.) **[PTA -3]**

Sol. Let $P(x) = ax^2 + bx + c$

Given $P(-3) = 21$

[$\because P(x) \div x + 3$, the remainder is 21]

$\Rightarrow a(-3)^2 + b(-3) + c = 21$

$\Rightarrow 9a - 3b + c = 21$ ...(1)

Also, $P(5) = 61$

$\Rightarrow a(5)^2 + b(5) + c = 61$

[using remainder theorem]

$\Rightarrow 25a + 5b + c = 61$ ...(2)

and $P(1) = 9$

$\Rightarrow a(1)^2 + b(1) + c = 9$

$\Rightarrow a + b + c = 9$ ...(3)

Reducing the augment matrix to an equivalent row-echelon form using elementary row operations, we get

$$\left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 25R_1]{R_3 \to R_3 - 9R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 4]{R_2 \to R_2 \div 4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & -3 & -2 & -15 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{3}{5}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & \frac{8}{5} & \frac{48}{5} \end{array}\right]$$

$$\xrightarrow{R_3 \to 5R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & 8 & 48 \end{array}\right]$$

Writing the equivalent equations from the row-echelon matrix we get,

$a + b + c = 9$ ...(1)

$-5b - 6c = -41$ ...(2)

$8c = 48$

$\Rightarrow c = \frac{48}{8} = 6$

Substituting $c = 6$ in (2) we get,

$\Rightarrow -5b - 6(6) = -41$

$\Rightarrow -5b = 36 - 41$

$\Rightarrow -5b = -41 + 36 = -5$

$\Rightarrow b = \frac{-5}{-5} = 1$

Substituting $b = 1$, $c = 6$ in (1) we get,

$a + 1 + 6 = 9$

$\Rightarrow a + 7 = 9$

$\Rightarrow a = 9 - 7$

$\Rightarrow a = 2$

$\therefore a = 2, b = 1$ and $c = 6$

3. An amount of ₹ 65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is ₹ 4,800. The income from the third bond is ₹ 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Sol. Let the price of bond invested in 6%, 8% and 9% rates be let ₹ x, ₹ y and ₹ z respectively

$\therefore$ By the given data, $x + y + z = 65{,}000$ ...(1)

$$\frac{6\times x\times 1}{100}+\frac{8\times y\times 1}{100}+\frac{9\times z\times 1}{100} = 4{,}800$$

$$\left[\because \text{Interest} = \frac{\text{PNR}}{100}\right]$$

$$\Rightarrow \frac{6x}{100}+\frac{8y}{100}+\frac{9z}{100} = 4{,}800$$

$$\Rightarrow 6x + 8y + 9z = 4{,}80{,}000 \quad ...(2)$$

$$\text{Also, } \frac{9z}{100} = 600 + \frac{8y}{100}$$

$$\Rightarrow \frac{-8y}{100}+\frac{9z}{100} = 600$$

$$\Rightarrow -8y + 9z = 60{,}000 \quad ...(3)$$

Reducing the augmented matrix to an equivalent row-echelon form by using elementary row operation, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 65{,}000 \\ 6 & 8 & 9 & 4{,}80{,}000 \\ 0 & -8 & 9 & 60{,}000 \end{array}\right]$$

$$\xrightarrow{R_2\to R_2-6R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65{,}000 \\ 0 & 2 & 3 & 90{,}000 \\ 0 & -8 & 9 & 60{,}000 \end{array}\right]$$

$$\xrightarrow{R_3\to R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65{,}000 \\ 0 & 2 & 3 & 90{,}000 \\ 0 & 0 & 21 & 4{,}20{,}000 \end{array}\right]$$

Writing the equivalent from the row echelon matrix we get,

$$x + y + z = 65{,}000 \quad ...(1)$$

$$2y + 3z = 90{,}000 \quad ...(2)$$

$$21z = 4{,}20{,}000$$

$$\Rightarrow z = \frac{4{,}20{,}000}{21} = 20{,}000$$

Substituting z = 20,000 in (2),

$$2y + 3(20{,}000) = 90{,}000$$

$$\Rightarrow 2y + 60{,}000 = 90{,}000$$

$$\Rightarrow 2y = 90{,}000 - 60{,}000 = 30{,}000$$

$$\Rightarrow y = \frac{30{,}000}{2} = 15{,}000$$

Substituting $y = 15{,}000$ and $z = 20{,}000$ in (1) we get,

$$x + 15{,}000 + 20{,}000 = 65{,}000$$

$$\Rightarrow x + 35{,}000 = 65{,}000$$

$$\Rightarrow x = 65{,}000 - 35{,}000$$

$$\Rightarrow x = 30{,}000$$

Thus the price of 6% bond is ₹ 30,000 the price of 8% bond is ₹ 15,000 and the price of 9% bond is ₹ 20,000.

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points (–6, 8),(–2 – 12) and (3,8). He wants to meet his friend at P(7,60). Will he meet his friend? (Use Gaussian elimination method.)

Sol. Given $y = ax^2 + bx + c$ ...(1)

(–6, 8) lies on (1)

$$\Rightarrow 8 = a(-6)^2 + b(-6) + c$$

$$\Rightarrow 8 = 36a - 6b + c \quad ...(2)$$

(–2, –12) lies on (1)

$$\Rightarrow -12 = a(-2)^2 + b(-2) + c$$

$$\Rightarrow -12 = 4a - 2b + c \quad ...(3)$$

Also (3, 8) lies on (1)

$$\Rightarrow 8 = a(3)^2 + b(3) + c$$

$$\Rightarrow 8 = 9a + 3b + c \quad ...(4)$$

Reducing the augment matrix to an equivalent row-echelon form by using elementary row operations, we get,

$$\left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right] \xrightarrow[R_3\to 4R_3-R_1]{R_2\to 9R_2-R_1} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right]$$

$$\xrightarrow[R_3\to R_3 \div 3]{R_2\to R_2 \div 4} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{R_3\to R_3+2R_2} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right]$$

Writing the equivalent equation from the row echelon matrix, we get $36a - 6b + c = 8$ ...(1)

$$-3b + 2c = -29 \quad ...(2)$$

$$5c = -50$$

$$\Rightarrow c = \frac{-50}{5} = -10$$

Substituting $c = -10$ in (2) we get,

$$-3b + 2(-10) = -29$$
$$\Rightarrow -3b - 20 = -29$$
$$\Rightarrow -3b = -29 + 20$$
$$\Rightarrow -3b = -9$$
$$\Rightarrow b = \frac{-9}{-3} = 3$$

Substituting $b = 3$ and $c = -10$ in (1) we get,

$$36a - 6(3) - 10 = 8$$
$$\Rightarrow 36a - 18 - 10 = 8$$
$$\Rightarrow 36a - 28 = 8$$
$$\Rightarrow 36a = 8 + 28 = 36$$
$$\Rightarrow a = \frac{36}{36} = 1$$
$$\therefore a = 1, b = 3, c = -10$$

Hence the path of the boy is

$$y = 1(x^2) + 3(x) - 10$$
$$\Rightarrow y = x^2 + 3x - 10$$

Since his friend is at P(7, 60),

$$60 = (7)^2 + 3(7) - 10$$
$$\Rightarrow 60 = 49 + 21 - 10$$
$$\Rightarrow 60 = 70 - 10 = 60$$
$$\Rightarrow 60 = 60$$

Since (7, 60) satisfies his path , he can meet his friend who is at P(7, 60)

EXERCISE 1.8

Choose the Correct or the most suitable answer from the given four alternatives :

1. If $|\text{adj (adj A)}| = |A|^9$, then the order of the square matrix A is

(1) 3 (2) 4 (3) 2 (4) 5

[Ans. (2) 4]

Hint : $|\text{adj (adj)A}| = |A|^{(n-1)^2}$

$$\therefore (n-1)^2 = 9 \Rightarrow (n-1)^2 = 3^2$$
$$\Rightarrow n - 1 = 3 \Rightarrow n = 4$$

2. If A is a 3 × 3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then $BB^T =$

(1) A (2) B (3) I_3 (4) B^T

[Ans. (3) I_3]

Hint : $BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$

$$= (A^{-1}A^T)(A^T)^T \cdot (A^{-1})^T$$
$$= (A^{-1}A^T)A(A^{-1})^T = A^{-1}(A.A^T)(A^{-1})^T$$
$$= (A^{-1}A).A^T(A^T)^{-1} \quad [\because (A^{-1})^T = (A^T)^{-1}]$$
$$= I \cdot I = I \quad [\because A^T \cdot (A^T)^{-1} = I]$$

3. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, then $\frac{|\text{adj B}|}{|C|} =$ [Govt. MQP-2019]

(1) $\frac{1}{3}$ (2) $\frac{1}{9}$ (3) $\frac{1}{4}$ (4) 1

[Ans. (2) $\frac{1}{9}$]

Hint : $\frac{|\text{adj}B|}{|C|} = \frac{|\text{adj(adj A)}|}{|3A|} = \frac{|A|^{(n-1)^2}}{|3|^2.|A|}$

$$= \frac{|A|^2}{9 \cdot |A|} = \frac{|A|}{9.|A|} = \frac{1}{9}$$

4. If $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then A =

(1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

(3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

[Ans. (3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$]

Hint : $AX = B$

$\Rightarrow A = BX^{-1}$ where

$$X = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \cdot \frac{1}{|X|} \cdot \text{adj (X)}$$

$$= \not{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\not{6}}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then $9I_2 - A =$ [PTA - 2]

(1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ (4) $2A^{-1}$

[Ans. (4) $2A^{-1}$]

Hint : $9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 9-7 & 0-3 \\ 0-4 & 9-2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \text{adj A}$$

But $A^{-1} = \frac{1}{|A|}\text{adj A} = \frac{1}{2}\text{adj A} \Rightarrow \text{adj A} = 2A^{-1}$

6. **If** $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ **then |adj (AB)| =**

(1) –40 (2) –80 (3) –60 (4) –20

[Ans. (2) –80]

Hint :

$$AB = \begin{bmatrix} 2+0 & 8+0 \\ 1+10 & 4+0 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 11 & 4 \end{bmatrix}$$

$$\text{adj}(AB) = \begin{bmatrix} 4 & -8 \\ -11 & 2 \end{bmatrix}$$

$$|\text{adj}(AB)| = 8 - 88 = -80$$

7. **If** $P = \begin{vmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{vmatrix}$ **is the adjoint of 3 × 3 matrix A and |A| = 4, then *x* is**

(1) 15 (2) 12 (3) 14 (4) 11

[Ans. (4) 11]

Hint : $|\text{adj } A| = |A|^{n-1}$

$$1\begin{vmatrix} 3 & 0 \\ 4 & -2 \end{vmatrix} - x\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} + 0 = 4^{3-1}$$

$$\Rightarrow -6 - x(-2) = 4^2 \quad \Rightarrow \quad -6 + 2x = 16$$

$$\Rightarrow \quad 2x = 22 \quad \Rightarrow \quad x = 11$$

8. **If** $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ **and** $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ **then the value of** a_{23} **is** **[PTA - 5]**

(1) 0 (2) –2 (3) –3 (4) –1

[Ans. (4) –1]

Hint :

$$|A| = 3\begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= 3(2) - 1(-2) - 1(4+2)$$

$$= 6 + 2 - 6 = 2$$

$$a_{23} = \frac{1}{|A|}\text{ co-factor of } a_{32} = \frac{1}{2} \times -\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

$$= -\frac{1}{2}(0+2) = \frac{-2}{2} = -1$$

9. **If A, B and C are invertible matrices of some order, then which one of the following is not true?**

(1) adj A = $|A|A^{-1}$

(2) adj (AB) = (adj A) (adj B)

(3) det A^{-1} = $(\det A)^{-1}$

(4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

[Ans. (2) adj (AB) = (adj A) (adj B)]

10. **If** $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$ **and** $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, **then** B^{-1} = **[Mar. - 2020]**

(1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ (2) $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

[Ans. (1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$**]**

Hint : Since $(AB)^{-1} = B^{-1}A^{-1}$, we get,

$$\begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} = B^{-1}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$\text{Let } X = B^{-1}Y$$

$$B^{-1} = XY^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} \cdot \frac{1}{|Y|}.(\text{adj } Y)$$

$$= \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} \cdot \frac{1}{1}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36-34 & 12-17 \\ -57+54 & -19+27 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$$

11. **If** $A^T. A^{-1}$ **is symmetric, then** A^2 = **[Sep. - 2020]**

(1) A^{-1} (2) $(A^T)^2$

(3) A^T (4) $(A^{-1})^2$

[Ans : (2) $(A^T)^2$**]**

Hint :

$$\Rightarrow \quad A^T A^{-1} = (A^T A^{-1})^T$$

$$= (A^{-1})^T (A^T)^T = (A^{-1})^T A$$

$$\Rightarrow \quad A^T A^{-1} = (A^{-1})^T. A$$

$$\Rightarrow \quad A^T A^{-1} = (A^T)^{-1}. A$$

[Premultiplying by A^T and post multiplying by A on both sides, we get]

$$A^T(A^T A^{-1})A = A^T[(A^T)^{-1}. A]A$$

$$\Rightarrow \quad (A^T)^2(A^{-1}A) = A^T(A^T)^{-1}. A^2$$

$$\Rightarrow \quad (A^T)^2(I) = I.A^2$$

$$\Rightarrow \quad (A^T)^2 = A^2$$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1}$ = [PTA -1 ;Hy - 2019]

(1) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$

(3) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ (4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

[Ans : (4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$]

Hint : $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

13. If $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is

(1) $\frac{-4}{5}$ (2) $\frac{-3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

[Ans: (1) $\frac{-4}{5}$]

Hint : Since $A^T = A^{-1}$, $AA^T = A^T A = I$ [∴ they are orthogonal]

$$\Rightarrow \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}\begin{bmatrix} \frac{3}{5} & x \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{9}{25}+\frac{16}{25} & \frac{3x}{5}+\frac{12}{25} \\ \frac{3x}{5}+\frac{12}{25} & x^2+\frac{9}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{3x}{5}+\frac{12}{25} = 0 \text{ [Equating } a_{12} \text{ both sides]}$$

$$\Rightarrow \frac{3x}{5} = \frac{-12}{25} \Rightarrow x = \frac{-12}{25} \times \frac{5}{3} = \frac{-4}{5}$$

14. If $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and $AB = I_2$, then B = [PTA -3]

(1) $\left(\cos^2\frac{\theta}{2}\right)A$ (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$

(3) $(\cos^2\theta)I$ (4) $\left(\sin^2\frac{\theta}{2}\right)A$

[Ans: (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$]

Hint : $B = A^{-1} = \frac{1}{|A|}\text{adj } A$

$$= \frac{1}{1+\tan^2\frac{\theta}{2}}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} = \frac{1}{\sec^2\frac{\theta}{2}}.A^T = \left(\cos^2\frac{\theta}{2}\right)A^T$$

15. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A(\text{adj } A) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then k =

(1) 0 (2) sin θ (3) cos θ (4) 1

[Ans: (4) 1]

Hint : We know $A(\text{adj } A) = (\text{adj } A)A = |A|\, I$

$\Rightarrow |A| = k$

$$\therefore k = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

16. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $\lambda A^{-1} = A$, then λ is

(1) 17 (2) 14 (3) 19 (4) 21

[Ans: (3) 19]

Hint : $\lambda.\frac{1}{|A|}\text{adj } A = A$

$$\Rightarrow \lambda.\frac{1}{(-4-15)}.\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \frac{-\lambda}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \frac{\lambda}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \frac{\lambda}{19} = 1 \Rightarrow \lambda = 19$$

17. If $\text{adj } A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $\text{adj } B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then adj (AB) is [Hy - 2019]

(1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$

(3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

[Ans: (2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$]

Hint : adj (AB) = (adj B) (adj A)

$$= \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2-8 & 3+2 \\ -6+4 & -9-1 \end{bmatrix} = \begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ **is** **[Qy-2019]**

(1) 1 (2) 2 (3) 4 (4) 3

[Ans: (1) 1]

Hint : $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rank is 1 [$\because$ only one non – zero row]

19. If $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$**, then the values of** x **and** y **are respectively,** **[PTA -3; Sep. - 2020]**

(1) $e^{(\Delta_2/\Delta_1)}, e^{(\Delta_3/\Delta_1)}$

(2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$

(3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$

(4) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$ **[Ans: (4)** $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$**]**

Hint :

$$x^a y^b = e^m$$

$$\Rightarrow a \log x + b \log y = m$$

[Taking log both sides]

$$x^c . y^d = e^n$$

$$\Rightarrow c \log x + d \log y = n$$

$$\text{put } \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$$

$$\log x = \frac{\Delta_1}{\Delta_3} \text{ and } \log y = \frac{\Delta_2}{\Delta_3}$$

$$\therefore x = e^{\Delta_1/\Delta_3} \text{ and } y = e^{\Delta_2/\Delta_3}$$

20. Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) If A is a square matrix of order n and λ is a scalar, then adj $(\lambda A) = \lambda^n$ adj (A).

(iv) A(adjA) = (adj A) A = |A|I

(1) Only (i) (2) (ii) and (iii)

(3) (iii) and (iv) (4) (i), (ii) and (iv)

[Ans: (4) (i) (ii) and (iv)]

21. If ρ(A) = ρ ([A|B]), then the system AX = B of linear equations is **[PTA-6; Mar. - 2020]**

(1) consistent and has a unique solution

(2) consistent

(3) consistent and has infinitely many solution

(4) inconsistent **[Ans: (2) Consistent]**

22. If $0 \le \theta \le \pi$ **and the system of equations** $x + (\sin\theta)y - (\cos\theta)z = 0, (\cos\theta)x - y + z = 0, (\sin\theta)x + y - z = 0$ **has a non-trivial solution then θ is** **[Qy-2019]**

(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{4}$

[Ans: (4) $\frac{\pi}{4}$ **]**

Hint : $A = \begin{bmatrix} 1 & \sin\theta & -\cos\theta \\ \cos\theta & -1 & 1 \\ \sin\theta & 1 & -1 \end{bmatrix}$

The system has non – trivial solution if |A| = 0

$$\begin{vmatrix} 1 & \sin\theta & -\cos\theta \\ \cos\theta & -1 & 1 \\ \sin\theta & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \sin\theta\begin{vmatrix} \cos\theta & 1 \\ \sin\theta & -1 \end{vmatrix} - \cos\theta\begin{vmatrix} \cos\theta & -1 \\ \sin\theta & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(1 - 1) - \sin\theta\,(-\cos\theta - \sin\theta) - \cos\theta\,(\cos\theta + \sin\theta) = 0$

$\Rightarrow \sin\theta \cancel{\cos\theta} + \sin^2\theta - \cos^2\theta - \cancel{\cos\theta} \sin\theta = 0$

$\Rightarrow \cos^2\theta - \sin^2\theta = 0 \Rightarrow \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2}$ $\left[\because \cos\frac{\pi}{2} = 0\right]$

$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda-7 & \mu+5 \end{bmatrix}$**. The system has infinitely many solutions if**

(1) $\lambda = 7, \mu \ne -5$ (2) $\lambda = -7, \mu = 5$

(3) $\lambda \ne 7, \mu \ne -5$ (4) $\lambda = 7, \mu = -5$

[Ans: (4) $\lambda = 7, \mu = -5$**]**

Hint : When $\lambda = 7$ and $\mu = -5$,

$$[A|B] = \begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ρ(A) = ρ([A|B]) = 2 < 3, the number of unknowns.

$\therefore$ The system is consistent and has infinitely many solutions.

24. Let $A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B=\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is

(1) 2 (2) 4 (3) 3 (4) 1

[Ans: (4) 1]

Hint : $A = B^{-1} \Rightarrow A.B = B^{-1}.B \Rightarrow AB = I$

$$\therefore \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{4}a_{13} = 0$$

$$\Rightarrow \frac{1}{4}.[-2-x+3] = 0$$

$$\Rightarrow -x+1 = 0 \times 4 = 0$$

$$\Rightarrow x = 1$$

25. If $A=\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then adj (adj A) is **[PTA - 2]**

(1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

[Ans: (1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$]

Hint : $\text{adj (adj A)} = |A|^{n-2} A = |A|.A$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}$$

$$= 3(-3+4)+3(2-0)+4(-2-0)$$

$$= 3(1)+6-8=1$$

$$\therefore \text{adj (adj A)} = 1.A = A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

PTA QUESTION & ANSWERS

1 MARK

1. If A and B are orthogonal, then $(AB)^T (AB)$ is **[PTA - 1]**

(1) A (2) B (3) I (4) A^T

[Ans: (3) I]

Hint :

$$AA^T = A^TA = I$$
$$BB^T = B^TB = I$$
$$(AB)^T(AB) = B^TA^T(AB) = B^T(A^TA)B$$
$$= B^T(IB) = I$$

2. The adjoint of 3×3 matrix P is $\begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$, then the possible value(s) of the determinant P is (are) **[PTA - 4]**

(1) 3 (2) –3 (3) ± 3 (4) $\pm\sqrt{3}$

[Ans: (3) ±3]

Hint : $\begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -1[1-4]-2[1-4]+2[2-2]$

$$= -1(-3)-2(-3)+0 = 3+6 = 9$$

$$|P| = \pm\sqrt{9} = \pm 3$$

3. If A is a 3 × 3 matrix such that $|3\,adj\,A| = 3$ then |A| is equal to **[PTA - 5]**

(1) $\frac{1}{3}$ (2) $-\frac{1}{3}$ (3) $\pm\frac{1}{3}$ (4) ± 3

[Ans: (3) $\pm\frac{1}{3}$]

Hint : $|3\,adj\,A| = 27\,|adj\,A|$

$$3 = 27|A|^2$$

$$|A|^2 = \frac{1}{9}$$

$$|A| = \pm\frac{1}{3}$$

4. Let A be a non-singular matrix then which one of the following is false **[PTA - 6]**

(1) $(adj A)^{-1} = \frac{A}{|A|}$

(2) I is an orthogonal matrix

(3) $adj(adjA) = |A|^n A$

(4) If A is symmetric then *adj* A is symmetric

[Ans: (3) $adj(adjA) = |A|^n A$]

Hint : $adj\,(adjA) = |A|^{n-2} A$

2 MARKS

1. Prove that $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ is orthogonal. [PTA - 1]

Sol. Let $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

Then, $A^T = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^T = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

$AA^T = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

$= \begin{bmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta\\ \sin\theta\cos\theta-\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta\end{bmatrix}$

$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I_2.$

Similarly, we get $A^TA = I_2$.

Hence $AA^T = A^TA = I_2 \Rightarrow$ A is orthogonal.

2. If $A = \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$, $B = \begin{bmatrix}4 & 0\\ 2 & 5\end{bmatrix}$, find *adj* (AB). [PTA - 3]

Sol. $AB = \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}4 & 0\\ 2 & 5\end{bmatrix} = \begin{bmatrix}8+6 & 0+15\\ 4+4 & 0+10\end{bmatrix} = \begin{bmatrix}14 & 15\\ 8 & 10\end{bmatrix}$

$adj\,(AB) = \begin{bmatrix}10 & -15\\ -8 & 14\end{bmatrix}$

3 MARKS

1. If $A = \begin{bmatrix}3 & -2\\ \lambda & -2\end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. [PTA - 2]

Sol. $A^2 = \begin{bmatrix}3 & -2\\ \lambda & -2\end{bmatrix}\begin{bmatrix}3 & -2\\ \lambda & -2\end{bmatrix}$

$= \begin{bmatrix}9-2\lambda & -6+4\\ 3\lambda-2\lambda & -2\lambda+4\end{bmatrix} = \begin{bmatrix}9-2\lambda & -2\\ \lambda & -2\lambda+4\end{bmatrix}$

$\lambda A - 2I = \lambda\begin{bmatrix}3 & -2\\ \lambda & -2\end{bmatrix} - 2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$= \begin{bmatrix}3\lambda & -2\lambda\\ \lambda^2 & -2\lambda\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}3\lambda-2 & -2\lambda\\ \lambda^2 & -2\lambda-2\end{bmatrix}$

$\therefore -2\lambda = -2$

$\lambda = 1$

2. Find the rank of the matrix $\begin{bmatrix}4 & -2 & 6 & 8\\ 1 & 1 & -3 & -1\\ 15 & -3 & 9 & 21\end{bmatrix}$ [PTA - 4]

Sol. $\begin{bmatrix}4 & -2 & 6 & 8\\ 1 & 1 & -3 & -1\\ 15 & -3 & 9 & 21\end{bmatrix}$

$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix}1 & 1 & -3 & -1\\ 4 & -2 & 6 & 8\\ 15 & -3 & 9 & 21\end{pmatrix}$

$\xrightarrow[R_3 \to R_3+(-15)R_1]{R_2 \to R_2+(4)R_1} \begin{bmatrix}1 & 1 & -3 & -1\\ 0 & -6 & 18 & 12\\ 0 & -18 & 54 & 36\end{bmatrix}$

$\xrightarrow[R_3 \to \frac{R_3}{-18}]{R_2 \to \frac{R_2}{-6}} \begin{bmatrix}1 & 1 & -3 & -1\\ 0 & 1 & -3 & -2\\ 0 & 1 & -3 & -2\end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix}1 & 1 & -3 & -1\\ 0 & 1 & -3 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$

3. Solve by matrix inversion method : $5x + 2y = 4$, $7x + 3y = 5$. [PTA - 5]

Sol. Matrix form

$AX = B$

$A = \begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}$, $X = \begin{bmatrix}x\\ y\end{bmatrix}$, $B = \begin{bmatrix}4\\ 5\end{bmatrix}$

$|A| = \begin{vmatrix}5 & 2\\ 7 & 3\end{vmatrix} = 15 - 14 = 1$

$A^{-1} = \begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}$

$X = A^{-1}B$

$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}\begin{bmatrix}4\\ 5\end{bmatrix}$

$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}12 & -10\\ -28 & +25\end{bmatrix} = \begin{bmatrix}2\\ -3\end{bmatrix}$

$x = 2, y = -3$

4. Find the adjoint of the matrix $A = \begin{bmatrix}1 & 3\\ 2 & -5\end{bmatrix}$ and verify that A(*adj*A) = (*adj*A)A = |A|I. [PTA - 6]

Sol. $A = \begin{bmatrix}1 & 3\\ 2 & -5\end{bmatrix}$

$adj\,A = \begin{bmatrix}-5 & -3\\ -2 & 1\end{bmatrix}$

$A(adjA) = \begin{bmatrix}1 & 3\\ 2 & -5\end{bmatrix}\begin{bmatrix}-5 & -3\\ -2 & 1\end{bmatrix}$

$= \begin{bmatrix}-5-6 & -3+3\\ -10+10 & -6-5\end{bmatrix} = \begin{bmatrix}-11 & 0\\ 0 & -11\end{bmatrix}$ ...(1)

$$(\text{adj } A)\, A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5-6 & -15+15 \\ -2+2 & -6-5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} \quad ...(2)$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & -5 \end{vmatrix} = -5-6 = -11$$

$$|A|I = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} \quad ...(3)$$

From (1), (2) and (3)

$A(adjA) = (adj\, A)A = |A|I$

5 MARKS

1. Find the inverse of the non-singular matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, by elementary transformations. [PTA – 6]

Applying Gauss-Jordan method, we get

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \to 2R_1]{R_2 \to R_2 \to 3R_1} \left[\begin{array}{ccc|ccc} 1 & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & 0 & 0 \\ 3 & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & -\left(\frac{3}{2}\right) & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to 2R_2} \begin{bmatrix} 1 & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R + R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\text{So, } A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

GOVT. EXAM QUESTION & ANSWERS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. If the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}^n$ is $\frac{1}{11}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the ascending order of a, b, c, d is [Govt. MQP-2019]

(1) a, b, c, d (2) d, b, c, a

(3) c, a, b, d (4) b, a, c, d

[Ans: (2) d, b, c, a**]**

Hint : Inverse matrix $= \frac{1}{-5-6}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

$$= \frac{-1}{11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

2. If A is an orthogonal matrix, then |A| is [Qy - 2019]

(1) 1 (2) –1 (3) ± 1 (4) 0

[Ans: (3) ± 1**]**

Hint : The determinant of an orthogonal matrix is equal to 1 or –1.

2 MARKS

1. Solve the following system of linear equations by Cramer's rule $2x - y = 3$, $x + 2y = -1$. [Govt. MQP-2019]

Sol.

$2x - y = 3$

$x + 2y = -1$

$$\Delta = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \times 2 - (1)(-1) = 4 + 1 = 5$$

$$\Delta_1 = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 3 \times 2 - (-1) \times -1 = 6 - 1 = 5$$

$$\Delta_2 = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2 - 3 = -5$$

$$x = \frac{\Delta_1}{\Delta} = \frac{5}{5} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-5}{5} = -1$$

2. If adj A = $\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}. [Qy - 2019]

Sol. We compute |adj A| = 9

So, we get $A^{-1} = \pm \dfrac{1}{\sqrt{|adj\ A|}}$ adj (A)

$$= \pm \frac{1}{\sqrt{9}} \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \pm \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

3 MARKS

1. Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$ [Sep.- 2020]

Sol. We get

$$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{(0+6)}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \quad ...(1)$$

$$A^{-1} = \frac{1}{(0+3)}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{(2-0)}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4-3 & -3+0 \\ 0+2 & 0+0 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \quad ...(2)$$

As the matrices in (1) and (2) are same, $(AB)^{-1} = B^{-1}A^{-1}$ is verified.

5 MARKS

1. By using Gaussian elimination method, balance the chemical reaction equation: $C_5H_8 + O_2 \rightarrow CO_2 + H_2O$ [Qy - 2019]

Sol. We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$x_1\ C_5\ H_8 + x_2O_2 = x_3CO_2 + x_4H_2O$ ... (1)

The number of carbon atoms on the left-hand side of (1) should be equal to the number of carbon atoms on the right-hand side of (1). So we get a linear homogeneous equation

$$5x_1 = x_3$$
$$\Rightarrow \quad 5x_1 - x_3 = 0 \quad ... (2)$$

Similarly, considering hydrogen and oxygen atoms, we get respectively,

$$8x_1 = 2x_4$$
$$\Rightarrow \quad 4x_1 - x_4 = 0, \quad ... (3)$$
$$2x_2 = 2x_3 + x_4$$
$$\Rightarrow \quad 2x_2 - 2x_3 - x_4 = 0. \quad ... (4)$$

Equations (2), (3), and (4) constitute a homogeneous system of linear equations in four unknowns.

The augmented matrix is [A|B]

$$= \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

By Gaussian elimination method, we get

$$[A|B] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \rightarrow 4R_3 - 5R_1} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & -4 & 5 & 0 \end{array}\right]$$

Therefore, $\rho(A) = \rho([A|B]) = 3 < 4$ = Number of unknowns.

The system is consistent and has infinite number of solutions.

Writing the equations using the echelon form, we get $4x_1 - x_4 = 0$, $2x_2 - 2x_3 - x_4 = 0$, $-4x + 5x_4 = 0$.

So, one of the unknowns should be chosen arbitrarily as a non-zero real number.

Let us choose $x_4 = t$. Then, by back substitution, we $x_3 = \dfrac{5t}{4}$, $x_2 = \dfrac{7t}{4}$, $x_1 = \dfrac{t}{4}$

Since x_1, x_2, x_3 and x_4 are positive integers, let us choose $t = 4$.

Then, we get $x_1 = 1$, $x_2 = 7$, $x_3 = 5$ and $x_4 = 4$.

So, the balanced equation is

$C_5\ H_8 + 7O_2 \rightarrow 5CO_2 + 4H_2O$.