

PHYSICS

- ✦ MATHEMATICAL PHYSICS
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MATHEMATICAL PHYSICS

MATHEMATICAL PHYSICS

Dimensional Analysis

Definition:

Dimensional analysis is a method of checking the correctness of equations, converting units, and deriving relationships between physical quantities using their dimensions.

Fundamental Quantities:

1. Length $\rightarrow [L]$
2. Mass $\rightarrow [M]$
3. Time $\rightarrow [T]$
4. Temperature $\rightarrow [\Theta]$
5. Electric Current $\rightarrow [I]$
6. Luminous Intensity $\rightarrow [J]$
7. Amount of Substance $\rightarrow [N]$

Applications:

1. Checking dimensional consistency of equations.
2. Deriving relationships between physical quantities.
3. Converting units between different systems (CGS, MKS, SI).

Dimensional Homogeneity:

An equation is dimensionally correct if both sides of the equation have the same dimensions.

Example:

To check $v = u + at$:

- ☞ $[v] = [LT^{-1}]$, $[u] = [LT^{-1}]$, $[a] = [LT^{-2}]$, $[t] = [T]$
- ☞ So, $[at] = [LT^{-2}][T] = [LT^{-1}]$
- ☞ All terms have same dimensions $\Rightarrow$ Equation is dimensionally valid.

Limitations:

- ☞ Cannot give dimensionless constants.
- ☞ Cannot derive formula involving trigonometric, logarithmic or exponential functions.

Differential Equations

A. Ordinary Differential Equations (ODE)

Definition:

An equation involving the derivatives of a function with respect to a single variable.

Order of an ODE:

The order is the highest derivative in the equation.

Examples:

- ☞ $\frac{dy}{dx} + y = 0 \rightarrow$ 1st order
- ☞ $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0 \rightarrow$ 2nd order

Types:

1. First Order Linear: $\frac{dy}{dx} + P(x)y = Q(x)$
2. Second Order Linear: $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$

General Solution of First Order Linear ODE:

$$y(x) = e^{-\int P(x)dx} \left[\int Q(x)e^{\int P(x)dx} dx + C \right]$$

Example:

Solve: $\frac{dy}{dx} + y = e^x$

Solution:

- ☞ Integrating factor $= e^{\int 1 dx} = e^x$
- ☞ Multiply both sides: $e^x \frac{dy}{dx} + e^x y = e^{2x}$
- ☞ LHS $= \frac{dy}{dx} + (ye^x) = e^{2x}$
- ☞ Integrate: $ye^x = \frac{e^{2x}}{2} + C$
- ☞ Final: $y = \frac{1}{2}e^x + Ce^{-x}$

B. Partial Differential Equations (PDE)

Definition:

An equation involving partial derivatives of a function with two or more independent variables.

Order of a PDE:

Highest order of the partial derivative in the equation.

Examples:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \rightarrow \text{1st order}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow \text{2nd order}$$

Common Forms:

- ☞ Wave Equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
- ☞ Heat Equation: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$
- ☞ Laplace Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Vector Calculus: Gradient, Divergence, Curl

Vector Differential Operator: ∇ (Del)

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

A. Gradient (of scalar field ϕ)

$$\vec{\nabla} \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

- Points in the direction of maximum increase of the scalar field.
- Gradient is a vector.

Physical Meaning:

In thermodynamics, if ϕ = temperature, then $\nabla \phi$ is the temperature gradient.

B. Divergence (of vector field $\vec{A}$)

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- Measures the net outflow of a vector field from a point.
- Divergence is a scalar.

Example:

For an electric field $\vec{E}$, $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ (Gauss's Law)

C. Curl (of vector field $\vec{A}$)

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- Measures the rotational tendency of a vector field.
- Curl is a vector.

Example:

In magnetostatics: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's Law)

Important Vector Identities:

1. $\nabla \cdot (\nabla \times \vec{A}) = 0$
2. $\nabla \times (\nabla \phi) = 0$
3. $\nabla \cdot (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \phi)$
4. $\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + (\nabla \phi) \times \vec{A}$
5. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

Summary

Topic	Type	Key Formula / Concept
Dimensional Analysis	Conceptual	Dimensional Homogeneity
Ordinary Differential Eq.	Analytical	Order, Solution Techniques
Partial Differential Eq.	Analytical	Laplace, Heat, Wave Eq.
Gradient	Vector Field	$\nabla \phi$
Divergence	Vector Field	$\nabla \cdot \vec{A}$
Curl	Vector Field	$\nabla \times \vec{A}$

VECTOR CALCULUS: LAPLACIAN, THEOREMS, AND OPERATIONS

Vector Algebra Basics

Operation	Description	Formula
Scalar (Dot) Product	Gives a scalar	$\vec{A} \cdot \vec{B} = AB \cos \theta$
Vector (Cross) Product	Gives a vector	$\vec{A} \times \vec{B} = AB \sin \theta \hat{n}$
Triple Scalar Product	Volume of a parallelepiped	$\vec{A} \cdot (\vec{B} \times \vec{C})$
Triple Vector Product	Vector identity	$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Laplacian Operator

Definition:

The Laplacian of a scalar field ϕ is:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

It's a scalar that measures the "curvature" or "spread" of the scalar field.

Laplacian of a vector field:

$$\nabla^2 \vec{A} = \nabla^2 A_x \hat{i} + \nabla^2 A_y \hat{j} + \nabla^2 A_z \hat{k}$$

Applications:

- Appears in Poisson's Equation: $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$
- Heat Equation: $\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t}$

➤ Wave Equation: $\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$

Vector Integral Theorems

Gauss Divergence Theorem

Statement:

$$\oint_S \vec{A} \cdot \hat{n} dS = \int_V (\nabla \cdot \vec{A}) dV$$

- LHS: Surface integral (flux across closed surface).
- RHS: Volume integral of divergence.

Interpretation:

Total flux out of a closed surface = integral of divergence inside the volume.

Green's Theorem (in 2D)

Statement:

For a region R enclosed by curve C:

$$\oint_C (P dx + Q dy) = \iint_R \left(\frac{dQ}{dx} - \frac{\partial P}{\partial y} \right) dx dy$$

- Used in 2D problems.
- Special case of Gauss' theorem in the plane.

Stokes' Theorem

Statement:

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{S}$$

- LHS: Line integral of vector field along closed curve CCC.
- RHS: Surface integral of curl over surface SSS bounded by CCC.

Interpretation:

Circulation of a vector field = flux of its curl through the surface.

Key Vector Identities

Identity	Expression
$\nabla \cdot (\nabla \times \vec{A})$	0
$\nabla \times (\nabla \phi)$	0
$\nabla \cdot (\phi \vec{A})$	$\phi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \phi)$
$\nabla \times (\phi \vec{A})$	$\phi (\nabla \times \vec{A}) + (\nabla \phi) \times \vec{A}$

Summary

Topic	Result Type	Description
Gradient	Vector	Direction of maximum change
Divergence	Scalar	Outflow or source strength

Topic	Result Type	Description
Curl	Vector	Rotation or circulation
Laplacian (Scalar)	Scalar	Sum of second partial derivatives
Gauss Theorem	Surface ↔ Volume	Relates flux to divergence
Green's Theorem	Curve ↔ Area	2D equivalent of Gauss
Stokes' Theorem	Curve ↔ Surface	Circulation and curl relation

MATRICES

Types of Matrices

Matrix Type	Definition
Row Matrix	A matrix with only one row. Ex: [2 4 6]
Column Matrix	A matrix with only one column. Ex: $\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$
Square Matrix	Same number of rows and columns.
Diagonal Matrix	Non-zero elements only on the main diagonal.
Scalar Matrix	A diagonal matrix with equal diagonal elements.
Identity Matrix (I)	Diagonal elements = 1, rest = 0.
Zero Matrix	All elements are zero.
Symmetric Matrix	$A = A^T$
Skew-Symmetric	$A = -A^T$
Hermitian Matrix	$A = A^\dagger$, i.e., equal to its conjugate transpose.
Unitary Matrix	$A^\dagger A = AA^\dagger = I$
Orthogonal Matrix	$A^T A = AA^T = I$

Basic Matrix Operations

- Transpose: Flip rows into columns.
- Conjugate: Replace each element with its complex conjugate.
- Adjoint (Adj A): Transpose of the cofactor matrix.
- Inverse: $A^{-1} = \frac{1}{|A|} \cdot \text{Adj } |A| \neq 0$
- Trace: Sum of diagonal elements of a square matrix.

Rank of a Matrix

- The rank is the maximum number of linearly independent rows or columns.
- Echelon form is used to determine the rank.
- A matrix of rank r has at least one non-zero $r \times r$ minor, but all higher-order minors are zero.

Hermitian and Unitary Matrices

Matrix Type	Condition
Hermitian	$A = A^\dagger$
Skew-Hermitian	$A = -A^\dagger$
Unitary	$A^\dagger A = I$

Transformation of Matrices

- Used in changing basis or rotating coordinate systems.
- If X is a vector and A is a transformation matrix, the transformed vector Y is given by:
 $Y = AX$
- Orthogonal Transformation: Preserves the dot product and vector lengths.

Eigenvalues and Eigenvectors

- Let $AX = \lambda X$, where:
 - > A : Square matrix
 - > λ : Eigenvalue
 - > X : Eigenvector (non-zero)
- To find λ :
 $\text{Det}(A - \lambda I) = 0$ (Characteristic Equation)
- To find the eigenvector, solve:
 $(A - \lambda I)X = 0$

Characteristic Equation

- Equation obtained by:
 $|A - \lambda I| = 0$
- Roots of this equation are eigenvalues.

Cayley-Hamilton Theorem

- Statement: Every square matrix satisfies its own characteristic equation.

Example:

Let A be a 2×2 matrix with characteristic equation:
 $\lambda^2 - 5\lambda + 6 = 0$
 Then Cayley-Hamilton says:
 $A^2 - 5A + 6I = 0$

Diagonalization

A matrix A is diagonalizable if there exists an invertible matrix P such that:

$$P^{-1}AP = D$$

where D is a diagonal matrix with eigenvalues of A on the diagonal.

Steps:

1. Find eigenvalues λ from the characteristic equation.
2. Find eigenvectors corresponding to each λ .

3. Construct matrix P using eigenvectors as columns.
4. Compute $D = P^{-1}AP$

Table

Concept	Key Formula / Definition
Rank of a matrix	Max number of linearly independent rows/columns
Trace	Sum of diagonal elements
Characteristic Equation	(
Eigenvalue Equation	$AX = \lambda X$
Cayley-Hamilton Theorem	Matrix satisfies its own characteristic equation
Diagonalization	$P^{-1}AP = D$
Hermitian Matrix	$A = A^\dagger$
Unitary Matrix	$A^\dagger A = I$

POLYNOMIALS: HERMITE, BESSEL, AND LEGENDRE FUNCTIONS

Hermite Polynomials

Definition:

Hermite polynomials $H_n(x)$ arise in quantum mechanics, especially in the solution of the quantum harmonic oscillator.

Differential Equation:

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0$$

Generating Function:

$$G(x, t) = e^{-t^2 + 2xt} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

First Few Hermite Polynomials:

n	$H_n(x)$
0	1
1	$2x$
2	$4x^2 - 2$
3	$8x^3 - 12x$
4	$16x^4 - 48x^2 + 12$

Orthogonality:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^n n! \delta_{nm}$$

Recurrence Relations:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Bessel Functions

Definition:

Bessel functions $J_n(x)$ are solutions of Bessel's differential equation, which appears in problems with cylindrical symmetry (e.g., heat conduction in a cylinder, wave propagation).

Bessel's Equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

Bessel Function of First Kind $J_n(x)$:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n}$$

First Few Values:

n	$J_n(x)$ (series expansion up to 2 terms)
0	$1 - \frac{x^2}{4} + \dots$
1	$\frac{x}{2} - \frac{x^3}{16} + \dots$

Orthogonality:

For zeros $\alpha_{n,k}$ of $J_n(x)$:

$$\int_0^1 x J_n(\alpha_{n,m} x) J_n(\alpha_{n,k} x) dx = 0 \quad \text{for } m \neq k$$

Recurrence Relations:

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\frac{d}{dx} J_n(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)]$$

Legendre Polynomials

Definition:

Legendre polynomials $P_n(x)$ appear in solving Laplace's equation in spherical coordinates (e.g., electrostatics, gravity).

Legendre Differential Equation:

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

Generating Function:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Rodrigues' Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

First Few Legendre Polynomials:

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$

Orthogonality:

$$\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{nm}$$

Recurrence Relation:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Summary

Function	Equation Form	Orthogonality Interval	Uses
Hermite $H_n(x)$	$y'' - 2xy' + 2ny = 0$	$(-\infty, \infty)$	Quantum oscillator
Bessel $J_n(x)$	$x^2 y'' + xy' + (x^2 - n^2)y = 0$	$(0, 1)$ or with weights	Cylindrical waves
Legendre $P_n(x)$	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	$(-1, 1)$	Electrostatics (spheres)

SPECIAL FUNCTIONS: BETA AND GAMMA FUNCTIONS

Gamma Function $\Gamma(n)$

Definition:

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

It generalizes the factorial:

$$\Gamma(n+1) = n!$$

Important Properties:

1. Recursive relation:

$$\Gamma(n+1) = n\Gamma(n)$$

2. Special values:

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

➤ $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$

3. Relation to factorial:

$\Gamma(n+1) = n!$ for integer n

4. Gamma of half-integer:

$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{n!2^{2n}}\sqrt{\pi}$

Graphical Behavior:

- The Gamma function is continuous and increases rapidly for large n .
- It is undefined for zero and negative integers.

Beta Function $B(m, n)$

Definition:

$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ for $m, n > 0$

Relation with Gamma Function:

$B(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

Properties:

1. Symmetry:

$B(m, n) = B(n, m)$

2. Reduction:

$B(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$

Summary

Function	Integral Form	Key Relations
Gamma	$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$	$\Gamma(n+1) = n\Gamma(n), \Gamma(1/2) = \sqrt{\pi}$
Beta	$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$	$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

3. Integral of sine and cosine:

$\int_0^{\frac{\pi}{2}} \sin^{m-1} \theta \cos^{n-1} \theta d\theta = \frac{1}{2} B\left(\frac{m}{2}, \frac{n}{2}\right)$

Comparison of Gamma and Beta Functions

Property	Gamma $\Gamma(n)$	Beta $B(m, n)$
Definition	$\int_0^\infty x^{n-1} e^{-x} dx$	$\int_0^1 x^{m-1} (1-x)^{n-1} dx$
Domain	$n > 0$	$m, n > 0$
Symmetry	No	$B(m, n) = B(n, m)$
Relation	Recursive: $\Gamma(n+1) = n\Gamma(n)$	$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
Applications	Quantum physics, thermodynamics, statistics	Probability theory, statistics, special functions

Example 1:

Evaluate $\Gamma(4)$

Solution:

$\Gamma(4) = (4-1)! = 3! = 6$

Example 2:

Evaluate $B(2, 3)$

Solution using gamma relation:

$B(2, 3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{1 \cdot 2}{24} = \frac{1}{12}$

COMPLEX ANALYSIS, PROBABILITY & STATISTICS

Complex Analysis

1. Complex Numbers

A complex number is written as:

$z = a + ib$, where $i = \sqrt{-1}$

Real part: α , Imaginary part: b

2. De Moivre's Theorem

$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$

Useful for: Expressing powers/roots of complex numbers in polar form.

3. Functions of Complex Variable

- Complex variable: $z = x + iy$
- Function: $f(z) = u(x, y) + iv(x, y)$
- Analytic Function: Differentiable at all points in a region.

4. Differentiability

$f(z)$ is differentiable at point z if:

$\lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$ exists and is independent of path

5. Cauchy-Riemann Conditions

For $f(z) = u + iv$ to be analytic:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

6. Harmonic Functions

A function u is harmonic if:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If $f(z) = u + iv$ is analytic, then both u and v are harmonic.

7. Complex Integration

➤ For $f(z)$, complex integral is path dependent:

$$\oint_C f(z) dz = \oint_C (u + iv)(dx + idy)$$

➤ Contour Integral: When path is closed, denoted by $\oint_C f(z) dz$

8. Cauchy's Integral Theorem

If $f(z)$ is analytic within a closed contour C , then:

$$\oint_C f(z) dz = 0$$

9. Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

Applicable when $f(z)$ is analytic and a is inside the contour C .

10. Taylor Series

If $f(z)$ is analytic at point z_0 :

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

11. Laurent Series

If $f(z)$ has singularity at z_0 , Laurent series is:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

12. Singular Points

- Isolated Singularity: Function not analytic at a point but analytic in a neighborhood.
- Pole: Singularity where function becomes infinite.
- Removable Singularity: Limit exists but function not defined at that point.

13. Residue Theorem

If $f(z)$ has isolated singularities inside contour C , then:

$$\oint_C f(z) dz = 2\pi i \sum \text{Residues at poles inside } C$$

Probability and Statistics

Probability Definitions

Probability: Likelihood of an event occurring.

2. Addition Rule

If A and B are events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

3. Multiplication Rule

If A and B are independent:

$$P(A \cap B) = P(A) \cdot P(B)$$

If dependent:

$$P(A \cap B) = P(A) \cdot P(B | A)$$

4. Example Problem

If $P(A)=0.5$, $P(B)=0.4$, and $P(A \cap B) = 0.2$, find $P(A \cup B)$

$$P(A \cup B) = 0.5 + 0.4 - 0.2 = 0.7$$

Statistics

1. Mean (Average):

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

2. Median:

- Middle value when data is arranged in order.
- If even number of values, average of two middle terms.

3. Mode:

Most frequently occurring value.

4. Standard Deviation (SD):

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$$

Measures spread or dispersion in the data.

Summary

Concept	Key Formula / Definition
De Moivre's Theorem	$(\cos\theta + i\sin\theta)^n$
Cauchy-Riemann Equations	$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
Analytic Function	Differentiable + C-R equations satisfied
Residue Theorem	$\oint_C f(z) dz = 2\pi i \sum \text{Res}$
Probability Addition	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Mean	$\bar{x} = \frac{1}{n} \sum x_i$
Standard Deviation	$\sigma = \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2}$

FOURIER TRANSFORMS & LAPLACE TRANSFORMS

Fourier Transforms

Definition

The Fourier Transform of a function $f(x)$ is:

$$f(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Its Inverse Fourier Transform is:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{-ikx} dk$$

Fourier Transform of Derivatives

If $f(x) \rightarrow F(k)$, then:

$$\frac{d^n f}{dx^n} \rightarrow (ik)^n F(k)$$

Fourier Cosine and Sine Transforms

Cosine Transform:

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(kx) dx$$

Sine Transform:

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(kx) dx$$

Inverse formulas are similar with cosine or sine.

Convolution Theorem

If $f(x)$ and $g(x)$ have transforms $F(k)$ and $G(k)$, then:

$$f * g = \int_{-\infty}^{\infty} f(x-y)g(y)dy \Rightarrow F(k) \cdot G(k)$$

Applications:

a) Diffusion Equation (Heat equation):

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

- For infinite medium, apply full Fourier transform.
- For semi-infinite medium, use Fourier sine/cosine transforms.

b) Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Apply Fourier transform in space variable to reduce to ODE.

Laplace Transforms

Definition

The Laplace Transform of a function $f(t)$ is:

$$L[f(t)] = f(s) = \int_0^{\infty} e^{-st} f(t) dt$$

The Inverse Laplace Transform is:

$$L^{-1}[F(s)] = f(t)$$

Common Transforms

Function $f(t)$	Laplace Transform $F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2 + a^2}$
$\cos(at)$	$\frac{s}{s^2 + a^2}$
$\delta(t)$	1
$u(t-a)$ (unit step)	$\frac{e^{-as}}{s}$

Properties of Laplace Transform

Property	Formula
Linearity	$L[af + bg] = aF(s) + bG(s)$
First Derivative	$L[f'(t)] = sF(s) - f(0)$
Second Derivative	$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$
Integration	$L\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$
Shifting (Time)	$L[f(t-a)u(t-a)] = e^{-as}F(s)$

Laplace Transform of Derivatives and Integrals

- Useful for solving ODEs with initial conditions.
- Converts differential equation into algebraic equation in sss-domain.

Application - Laplace Equation

Problem: Potential in a semi-infinite strip or region.
Apply Laplace transform to space or time variable depending on boundary conditions, solve algebraically, then invert.

Summary

Concept	Fourier Transform	Laplace Transform
Domain	$x \in (-\infty, \infty)$	$t \in [0, \infty)$
Kernel	e^{-ikx}	e^{-st}
Inverse	Integration over k	Bromwich integral
Use	Heat, wave, signal processing	ODEs, circuits, systems
Key App	Convolution, spectral methods	Initial value problems

CLASSICAL MECHANICS

LAWS OF CONSERVATION

- To move a football at rest, someone must kick it. To throw a stone upwards, one has to give it an upward push. A breeze causes the branches of a tree to swing; a strong wind can even move heavy objects. A boat moves in a flowing river without anyone rowing it. Clearly, some external agency is needed to provide force to move a body from rest.
- Likewise, an external force is needed also to retard or stop motion. You can stop a ball rolling down an inclined plane by applying a force against the direction of its motion. In these examples, the external agency of force (hands, wind, stream, etc) is in contact with the object. This is not always necessary. A stone released from the top of a building accelerates downward due to the gravitational pull of the earth. A bar magnet can attract an iron nail from a distance. This shows that external agencies (e.g. gravitational and magnetic forces) can exert force on a body even from a distance.
- In short, a force is required to put a stationary body in motion or stop a moving body, and some external agency is needed to provide this force.

LAW OF CONSERVATION OF ENERGY

- It states that, "The energy can neither be created nor destroyed, though it can be transformed from one form into any of the forms, in which the energy can exist."
- From the above statement, it is clear, that no machine can either create or destroy energy, though it can only transform from one form into another. We know that the output of a machine is always less than the input of the machine. This is due to the reason that a part of the input is utilized in overcoming friction of the machine. This does not mean that this part of energy, which is used in overcoming the friction, has been destroyed. But it reappears in the form of heat energy at the bearings and other rubbing surfaces of the machine, though it is not available to us for useful work.

The above statement may be exemplified as below:

- (i). In an electrical heater, the electrical energy is converted into heat energy
- (ii). In an electric bulb, the electrical energy is converted into light energy
- (iii). In a dynamo, the mechanical energy is converted into electrical energy

LINEAR MOMENTUM AND ANGULAR MOMENTUM

The linear momentum or translational momentum is the product of the mass and velocity of an object.

It is a vector quantity, possessing a magnitude and a direction. If m is an object's mass and v is its velocity (also a vector quantity), then the object's momentum is:

$$\text{Linear momentum} = \text{Mass of an object} \times \text{Velocity of an object}$$

$$P = m v$$

Similarly,

The angular momentum or rotational momentum is the rotational equivalent of linear momentum. It is the total motion possessed by a rotating body and is expressed mathematically as:

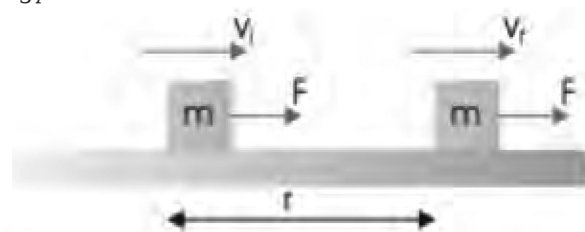
$$\text{Angular Momentum} = \text{Mass moment of inertia} \times \text{Angular velocity}$$

$$L = I \omega$$

WORK-ENERGY THEOREM

The work-energy theorem states that the work done by all forces acting on a particle equals the change in the particle's kinetic energy.

The principle of work and kinetic energy (also known as the work-energy theorem) states that the work done by the sum of all forces acting on a particle equals the change in the kinetic energy of the particle. This definition can be extended to rigid bodies by defining the work of the torque and rotational kinetic energy.



Kinetic Energy: A force does work on the block. The kinetic energy of the block increases as a result by the amount of work. This relationship is generalized in the work-energy theorem.

The work W done by the net force on a particle equals the change in the particle's kinetic energy (KE):

$$W = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

where v_i and v_f are the speeds of the particle before and after the application of force, and m is the particle's mass.

Derivation: For the sake of simplicity, we will consider the case in which the resultant force F is constant in both magnitude and direction and is parallel to the velocity of the particle. The particle is moving with constant acceleration a along a straight line. The